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Sinc-Galerkin method for solving biharmonic problems

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ABSTRACT

There are many techniques available to numerically solve the biharmonic equation. In this paper we show that the sinc-Galerkin method is a very effective tool in numerically solving this equation. Hermite interpolation is used to treat the nonhomogeneous boundary conditions. Our method is tested on examples and comparisons with other methods are made. It is shown that the sinc-Galerkin method yields good results even when singularities occur at the boundaries.

1. Introduction

Biharmonic problems play an important role in different scientific disciplines. They arise in several areas of mechanics of continua such as the stream function formulation for stationary Navier–Stokes equations [1], in two dimensional theory of elasticity [2,3], and the deformation of elastic plates [4]. They are encountered in the computation of the Airy stress function for plane stress problems [5] and in the steady Stokes flow of highly viscous fluids [6]. A more recent application of the biharmonic equation has been in the area of geometric and functional design, where it has been used to produce efficient mathematical descriptions of surfaces in physical space [7,8].

In this study we consider the biharmonic problem

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = F(x,y), \quad (x,y) \in D \tag{1.1}$$

subject to the homogeneous boundary conditions

$$u(x,y) = \frac{\partial u(x,y)}{\partial x} = 0, \quad \text{at} \quad x = a,b \quad \text{for} \quad c < y < d, \tag{1.2}$$

$$u(x,y) = \frac{\partial u(x,y)}{\partial y} = 0, \quad \text{at} \quad y = c,d \quad \text{for} \quad a < x < b. \tag{1.3}$$

where D is the rectangle $\{(x,y) : a < x < b, c < y < d\}$.

Many numerical methods have been developed for solving the biharmonic problem (1.1) numerically. Each of these methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more

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accurate methods is a continuous and ongoing process. Next, we present a selective review of the main and the most recent methods. These methods include: direct [9] and iterative solution of finite difference approximation [10–12]. Finite element treatment was used in [13] and finite volume schemes were used in [14]. Using boundary integral equation was given in [15]. The problem was reduced to Fredholm integral equation of the second kind by Mayo [16] to treat irregular boundaries and applied parallel processing to speed up the computation. A variational method was employed in [17]. A least squares method was given in [18]. Domian decomposition was used in [19]. Homotopy analysis method was used in [20]. Conformal mapping was used in [21]. Variation iteration was used by Ali and Raslan [22]. Mathematical programming techniques were used in [23].

Spectral techniques as: Legendre [24,25], integrated Chebyshev [26], Galerkin [27] and Sturm–Liouville collocation [28] were used. Also, using Chebyshev tau meshless method [29], polynomials and eigenfunction solutions [30] and the method of fundamental solutions [31]. Trefftz method was employed in [32], and Harr wavelets were used in [33,34].

In recent years, a lot of attention has been devoted to the study of the sinc methods to investigate various scientific models. It is possible to solve two point boundary value problems [35,36], initial-value problems [37], fourth-order differential equations [38], sixth order boundary-value problems [39], nonlinear higher-order boundary-value problems [40], advection–diffusion equation [41], partial differential equations [42], eigenvalue problems [43], singular problem-like poisson [44], linear Fredholm integro-differential equations [45], linear and nonlinear Volterra integro-differential equations [46], linear and nonlinear system of second-order boundary value problems [47], Troesch's problem [48] by using these methods. Stenger et al. [49] used sinc-convolution to treat the biharmonic problem. The comparison of finite difference, spectral and sinc-convolution treatments was considered in [50]. Very recently, El-Gamel and El-Hady introduced a comparison between the differential quadrature method and sinc method [51].

The outline of the paper is as follows. Section 2, contains notations, definitions and some results of sinc function theory. In Section 3, we use the sinc-Galerkin method to solve the biharmonic equations and obtain the discrete system. In Section 4, we introduce the method of solution of the discrete system. Section 5 presents appropriate techniques to treat nonhomogeneous boundary conditions. In Section 6, we give four numerical examples which will be tested to verify the reliability of the proposed algorithm.

2. Preliminaries and fundamentals

A general review of sinc function approximation is given in [52,53]. Hence, only properties important to the present goals are outlined in this section.

If $f(x)$ is defined on the real line, then for $h > 0$ the Whittaker cardinal expansion of f

$$f_m(x) = \sum_{k=-N}^{N} f_k S(k,h)(x), \quad m = 2N+1,$$

where $f_k = f(x_k)$, $x_k = hk$, and the mesh size is given by

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \quad 1 \leqslant \alpha \leqslant 2, \quad d \leqslant \frac{\pi}{2} \tag{2.1}$$

where N is suitably chosen and α depends on the asymptotic behavior of $f(x)$. The basis functions on (a,b) are then given by

$$S(k,h) \circ \phi(x) = \operatorname{sinc}\left(\frac{\phi(x) - kh}{h}\right)$$

and

$$\phi(x) = \ln\left(\frac{x-a}{b-x}\right). \tag{2.2}$$

The interpolation formula for $f(x)$ over $[a,b]$ takes the form

$$f(x) \approx \sum_{k=-N}^{N} f_k\, S(k,h) \circ \phi(x). \tag{2.3}$$

The nth derivative of the function f at points $x_k = (a + be^{kh})/(1 + e^{kh})$ can be approximated using a finite number of terms as

$$f^{(n)}(x_k) \cong h^{-n} \sum_{k=-N}^{N} \delta_{jk}^{(n)} f_k,$$

where

$$\delta_{jk}^{(n)} = \frac{d^n}{d\phi^n} S(j,h) \circ \phi(x)|_{x=x_k}.$$