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On semi-convergence of a class of Uzawa methods for singular saddle-point problems ☆

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ABSTRACT

In this paper, a class of Uzawa methods are presented for singular saddle-point problems. These methods contain the recently proposed Uzawa-AOR and Uzawa-SAOR methods as special cases. The (1,1)-block of the corresponding Uzawa preconditioner is positive definite. Both nonsingular and singular preconditioning matrices are considered. The semi-convergence of these methods is analyzed by using the techniques of singular value decomposition and Moore–Penrose inverse. Numerical results show that they need less workload per iteration step comparing with the GSOR and PIU methods in some situations, so they are feasible and effective for singular saddle-point problems in some situations.

1. Introduction

Consider saddle-point problems of the form

$$\mathscr{A}x \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix of $\text{rank}(B) = r < n$ with $m > n$, and $f \in \mathbb{R}^m, g \in \mathbb{R}^n$ are given vectors. Here, B^T denotes the transpose of B. Linear systems (1.1) are important and arise frequently in many different applications of scientific computing and engineering, including the fields of finite element methods for Navier–Stokes equations, constrained optimization, weighted linear least squares problems, image processing, linear elasticity, and so forth. See, for example [8,24,17,18,25].

When B is of full column rank, the linear system (1.1) is nonsingular. For this case, large varieties of effective iterative methods based on matrix splitting and their numerical properties have been proposed and discussed, such as the Uzawa-type methods [11,16,10,6,7,29,27,22] and the HSS methods [5,4,2,3,20]. In [29], Zhang and Shang considered the Uzawa-SOR method for nonsingular saddle-point problems, and Yun [27] generalized it to the Uzawa-AOR and Uzawa-SAOR methods. All these variants can be considered as inexact iteration methods with the Uzawa method as the outer iteration and the matrix relaxation method as the inner iteration.

However when B is rank deficient, (1.1) is singular. For this case, various kinds of relaxation iteration methods have also been established, especially the inexact Uzawa methods [32,30,28,21,22] and the HSS methods [1,13,26,14]. In this paper, we consider a class of special Uzawa methods, including the four variants in [29,27] as special cases, for singular saddle-point problems. The corresponding semi-convergence conditions are analyzed. Note that the related preconditioning matrices are

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both chosen as symmetric positive definite and symmetric positive semi-definite, which are different from some existent results for singular saddle-point problems.

Throughout this paper, we denote x^* as the conjugate transpose of the vector x. The minimum and maximum eigenvalues of a symmetric matrix H are denoted by $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$, respectively. I_n is the identity matrix of order n. null(A) denotes the null space of the matrix A. For two symmetric matrices E and F, $E \succeq F$ and $E \succ F$ denote that $E - F$ are positive semi-definite and positive definite, respectively.

The paper is organized as follows. In Section 2, a class of Uzawa methods for singular saddle-point problems are described and some basic concepts are proposed which will be used in the following parts. In Section 3 and 4,, semi-convergence of these methods are studied with nonsingular and singular preconditioning matrices, respectively. In Section 5, some algorithms of this class of Uzawa methods are given. In Section 6, a numerical example is used to examine these methods for solving singular saddle point problems. Finally, in Section 7, we give some conclusion remarks.

2. A class of Uzawa methods

In this section, we propose a class of Uzawa methods for singular saddle-point problems. Some basic concepts about semi-convergence are also given.

For the coefficient matrix $\mathcal{A}$ of (1.1), we make the splitting $\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U}$ with

$$\mathcal{D} = \begin{pmatrix} M & 0 \\ 0 & Q \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} M - A & -B \\ 0 & Q \end{pmatrix},$$

where $M \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are approximations of A and $B^TA^{-1}B$, respectively. Here $\omega A = M - N$ with ω being a nonzero real constant and M being positive definite, i.e., the symmetric part of M is positive definite. Denote

$$\Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix}$$

with τ also being a nonzero real constant. Further, we make matrix splitting $\mathcal{A} = \mathcal{M}(\omega,\tau) - \mathcal{N}(\omega,\tau)$ with the Uzawa preconditioner

$$\mathcal{M}(\omega,\tau) = \Omega^{-1}(\mathcal{D} - \Omega\mathcal{L}) = \begin{pmatrix} \frac{1}{\omega}M & 0 \\ -B^T & \frac{1}{\tau}Q \end{pmatrix} \tag{2.1}$$

and

$$\mathcal{N}(\omega,\tau) = \Omega^{-1}[(I_{m+n} - \Omega)\mathcal{D} + \Omega\mathcal{U}] = \begin{pmatrix} \frac{1}{\omega}N & -B \\ 0 & \frac{1}{\tau}Q \end{pmatrix}.$$

Thus we obtain a class of Uzawa methods with the following iteration scheme:

$$\begin{pmatrix} M & 0 \\ -\tau B^T & Q \end{pmatrix}\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \begin{pmatrix} N & -\omega B \\ 0 & Q \end{pmatrix}\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} + \begin{pmatrix} \omega f \\ -\tau g \end{pmatrix}. \tag{2.2}$$

Decompose A as $A = D - L - U$ with D being the diagonal part of A, $-L$ and $-U$ being the strict lower and strict upper parts of A, respectively. The parameters $\omega,\ s$ and τ used below are all positive real numbers.

Based on different selections of $\omega A = M - N$, we have the so-called the variants of Uzawa method as follows.

(1) The Uzawa-AOR method with

$$M = D - sL \equiv M_1(\omega,s) \quad \text{and} \quad N = (1-\omega)D + (\omega - s)L + \omega U \equiv N_1(\omega,s). \tag{2.3}$$

(2) The Uzawa-SAOR method with

$$M = (D - sL)C(\omega,s)^{-1}(D - sU) \equiv M_2(\omega,s), \tag{2.4}$$

$$N = ((1-\omega)D + (\omega - s)U + \omega L)C^{-1}(\omega,\tau)^{-1}((1-\omega)D + (\omega - s)L + \omega U) \equiv N_2(\omega,s).$$

Here $C(\omega,s) = (2-\omega)D + (\omega - s)(L+U)$. It can be seen that if $\omega = s$, the Uzawa-AOR and Uzawa-SAOR methods reduce to the Uzawa-SOR and Uzawa-SSOR methods, respectively.

As for the positive definiteness of $M_1(\omega,s)$ and $M_2(\omega,s)$ defined as (2.3) and (2.4), we have the following results.

Lemma 2.1. *If* $0 < \omega \leqslant s < 2$, *then*

(a) $M_1(\omega,s)$ *defined as* (2.3) *is positive definite;*
(b) $M_2(\omega,s)$ *defined as* (2.4) *is symmetric positive definite with* $C(\omega,s)$ *being symmetric positive definite.*