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On approximation of linear functionals over convex functions: Construction techniques and new directions

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ABSTRACT

Recently there has been renewed interest in the problem of finding under and over estimations on the set of convex functions to a given non-negative linear functional; that is, approximations which estimate always below (or above) the functional over a family of convex functions. The most important example of such an approximation problem is given by the multidimensional versions of the midpoint (rectangle) rule and the trapezoidal rule, which provide under and over estimations to the true value of the integral on the set of convex functions (also known as the Hermite–Hadamard inequality). In this paper, we introduce a general method of constructing new families of under/over-estimators on the set of convex functions for a general class of linear functionals. In particular, under the regularity condition, namely the functions belonging to $C^2(\Omega)$ (not necessarily convex), we will show that the error estimations based on such estimators are always controlled by the error associated with using the quadratic function. The result is also extended to the class of Lipschitz functions. We also propose a modified approximation technique to derive a general class of under/over estimators with better error estimates.

1. Some background and motivation

The problem of under and over-estimations of functionals in its most general form can be described in the following way: Let X be a real vector sub-space of real-valued continuous functions on Ω, a nonempty fixed compact convex set in $\mathbb{R}^d$, and T a linear functional on X. If a function f is given in X, it should be noted that in many situations, of course, the true value of $T[f]$ is, in general, impossible or not easy to compute, or often not know but only we can evaluate T on a given subset of X containing some simple functions. One popular numerical approach is to replace $T[f]$ by another simple approximating functional $A[f]$, which can relatively easy to evaluate numerically. We also wish to construct $A[f]$ with a possibly small error. The key idea to quantify the quality of the numerical approximation $A[f]$ is to use two different functionals, say $\underline{T}$ and $\overline{T}$, to estimate the absolute value of the error $|T[f] - A[f]|$ by $|\overline{T}[f] - \underline{T}[f]|$. If no other information is available, we are forced to accept this (or some scaling of it) as the error estimate of $|T[f] - A[f]|$. However, to get a good estimate of $T[f]$, we need some a priori informations about T in a given subset G of X (not necessarily containing f). The common practice in such a case is to construct lower and upper bounding functionals $\underline{T}$ and $\overline{T}$ in such a way that

$$\underline{T}[g] \leqslant T[g] \leqslant \overline{T}[g] \tag{1}$$

for any $g \in G$.

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A natural question to ask is therefore: Given a linear functional T on X and $f \in C(\Omega)$. Can we select appropriate functionals $A[f],\ \underline{T}[f],\ \overline{T}[f]$, and also decide how the deviation of $A[f]$ from $T[f]$ should measured?

Obviously, if we know that f belongs to the some set G, we can sometimes better evaluate and estimate the exact value of $T[f]$. Indeed, if that is the case, we may take the approximation functional $A := (1-\lambda)\underline{T} + \lambda\overline{T}$, any convex combination of $\underline{T}$ and $\overline{T}$ then, obviously, for any $\lambda \in [0,1]$, the error estimate can always be controlled as follows:

$$|T[f] - A[f]| \leqslant \overline{T}[f] - \underline{T}[f]. \tag{2}$$

Eq. (2), clearly, shows that when $\overline{T}[f] - \underline{T}[f]$ is small, we are confident that $|T[f] - A[f]|$ is also small. Hence, we are interested in solving the following under and over approximation problem.

Problem 1.1. For a given subset G of X, a linear functional T and a function $f \in X$ (no necessary in G,) how we can determine the functionals $\underline{T},\ \overline{T},\ A$ with $\underline{T} \leqslant T,\ A \leqslant \overline{T}$ in G and, in such a way that, we are able to control the error $|T[f] - A[f]|$?

Clearly, a basic issue of the success of this approach is in the ability to construct valid and rigorous under and over estimators. In order to make this idea more precise, a specific example may help to clarify most of the main points of this approach. We adopt the following notation: for a measurable set $D \subset \mathbb{R}^d$, we shall always denote by $|D|$ the measure of D. The set of all continuous convex functions defined on Ω will be denoted by $K(\Omega)$.

Example 1.2. As a concrete example, we take Ω a non-degenerate simplex in $\mathbb{R}^d$ with vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d$. Denote by $\boldsymbol{v}^*$ its center of gravity, which is defined by

$$\boldsymbol{v}^* = \frac{\int_\Omega \boldsymbol{x} d\boldsymbol{x}}{|\Omega|}.$$

For simplicity, let us consider the particularly important case when T is the normalized functional integral

$$T[f] = \frac{\int_\Omega f(\boldsymbol{x}) d\boldsymbol{x}}{|\Omega|},$$

which we may under and over estimate respectively by:

$$\underline{T}[g] = g(\boldsymbol{v}^*) \quad \text{and} \quad \overline{T}[g] = \frac{1}{d+1}\sum_{i=0}^{d} g(\boldsymbol{v}_i),$$

when we take G to be the set of all convex functions $K(\Omega)$. The verification of constraints hypothesis (1) of under and over estimations of $\underline{T}$ and $\overline{T}$ is based on the use of the multidimensional versions of the midpoint rule and the trapezoidal rule, see [15]. Indeed, T can be under and over estimated as follows:

$$\underline{T}[g] \leqslant T[g] \leqslant \overline{T}[g], \quad \text{for all} \quad g \in K(\Omega), \tag{3}$$

see [15, Theorems 2.1 and 2.2]. For a given $f \in C(\Omega)$, we now choose as an approximation to the true value of $T[f]$ any convex combination of $\underline{T}[f]$ and $\overline{T}[f]$. Then clearly $A[f]$ guarantees a better approximation than $\underline{T}[f]$ and $\overline{T}[f]$ in $K(\Omega)$, and for any λ, such that $0 \leqslant \lambda \leqslant 1/(d+1)$, the functional $A[f] = (1-\lambda)\underline{T}[f] + \lambda\overline{T}[f]$ systematically over estimates $T[f]$ in $K(\Omega)$, see [15, Theorem 4.1]. Moreover, as observed previously, under the assumption $f \in K(\Omega)$, we have the error estimate (2). However, if moreover f belongs to $C^2(\Omega)$, then for any λ, such that $0 \leqslant \lambda \leqslant 1/(d+1)$, we have the estimate, see [15, Corollary 6.5],

$$|T[f] - A[f]| \leqslant \frac{|D^2 f|_\Omega}{2}\left(\frac{1}{d+2} - \frac{\lambda}{d+1}\right)\sum_{i=0}^{d}\|\boldsymbol{v}_i - \boldsymbol{v}^*\|^2 \tag{4}$$

with $\|\cdot\|$ denoting the usual Euclidean norm for vectors in $\mathbb{R}^d$ and

$$|D^2 f|_\Omega := \sup_{\boldsymbol{x}\in\Omega} \sup\left\{\left|D^2_{\boldsymbol{y}} f(\boldsymbol{x})\right| \ : \ \boldsymbol{y} \in \mathbb{R}^d,\ \|\boldsymbol{y}\| = 1\right\}.$$

The above Problem 1.1 is obviously too general to be dealt with under a unifying aspect. From now on, we restrict ourselves to the case where the set $G = K(\Omega)$ and we are faced now with two issues: Given a linear functional T on X and $f \in C(\Omega)$.

(P1) Do there exist two linear functionals $\underline{T},\ \overline{T}$, that under and over estimate T as follows:

$$\underline{T}[g] \leqslant T[g] \leqslant \overline{T}[g], \quad \text{for all } g \in K(\Omega)? \tag{5}$$

(P2) Once we found an under and over estimator, how can we determine a ‘good’ approximation for $T[f]$?

In the case of functional integrals, the answer to the first issue has attracted the interest of people working in general inequalities, who refer to (5) as the Hermite–Hadamard inequalities (see [15]). We shall join them in doing so. In particular, we shall call the first and the second inequality in (5) the lower and the upper Hermite–Hadamard inequality, respectively. There is a big qualitative jump going from one to more dimensions when dealing with the Hermite–Hadamard inequalities