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Multiple nontrivial solutions for asymptotically linear discrete problem via computations of the critical groups ☆

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ABSTRACT

In this paper, we study the multiplicity of nontrivial solutions for asymptotically linear discrete problem by using critical point theory, Morse theory and minimax methods. We obtained that the problem has at least two nontrivial solutions under various conditions.

1. Introduction and main results

In this paper, we consider the multiplicity of solutions for the following discrete problem

$$\begin{cases} -\triangle^2 u(k-1) = f(k, u(k)), & k \in \mathbb{Z}[1,N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{1.1}$$

where N is a given positive integer with $N \geqslant 3$, $\mathbb{Z}[1,N] = \{1,2,\ldots,N\}$, $\triangle u(k) = u(k+1) - u(k)$, $\triangle^2 u(k) = \triangle(\triangle u(k))$, and $f(k,\cdot) \in C^1(\mathbb{R},\mathbb{R})$ for $k \in \mathbb{Z}[1,N]$. Assume that $f(k,0) = 0$ for $k \in \mathbb{Z}[1,N]$, then (1.1) admits a trivial solution $u = 0$. Therefore we are concerned on the existence of nontrivial solutions of (1.1). We assume that f has asymptotically linear property both at infinity and at zero, i.e., f satisfies

$$\lim_{|t|\to\infty} \frac{f(k,t)}{t} = \alpha_\infty(k), \quad k \in \mathbb{Z}[1,N] \tag{1.2}$$

and

$$\lim_{|t|\to 0} \frac{f(k,t)}{t} = \alpha_0(k), \quad k \in \mathbb{Z}[1,N], \tag{1.3}$$

where $\alpha_*(k),\ * = 0,\infty,\ k \in \mathbb{Z}[1,N]$ are fixed real numbers. Define

$$f_*(k,t) = f(k,t) - \alpha_*(k)t, \quad F_*(k,t) = \int_0^t f_*(k,s)ds, \quad * = 0,\infty,\ k \in \mathbb{Z}[1,N]. \tag{1.4}$$

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For the following linear eigenvalues problem

$$\begin{cases} -\Delta^2 u(k-1) - \alpha_*(k)u(k) = \lambda u(k), & k \in \mathbb{Z}[1,N], \\ u(0) = u(N+1) = 0 \end{cases} \tag{1.5}$$

from [1], we know that all eigenvalues of (1.5) are real and simple. Let the eigenvalues of (1.5) be

$$\lambda_1(*) < \lambda_2(*) < \cdots < \lambda_N(*)$$

and let the corresponding orthonormal eigenvectors be

$$\phi_1(*), \phi_2(*), \ldots, \phi_N(*).$$

Define

$$\mu_* = \#\{\lambda_k(*) | \lambda_k(*) < 0, \quad k \in \mathbb{Z}[1,N]\}, \quad \nu_* = \#\{\lambda_k(*) | \lambda_k(*) = 0, \quad k \in \mathbb{Z}[1,N]\}, \quad * = 0, \infty,$$

where $\#S$ denotes the number of elements in S.

In this paper, we make the following assumptions:

(f_∞^-) $\lim_{t\to+\infty} f_\infty(k,t) = -\infty, \quad \lim_{t\to-\infty} f_\infty(k,t) = +\infty, \quad k \in \mathbb{Z}[1,N],$

(f_∞^+) $\lim_{t\to+\infty} f_\infty(k,t) = +\infty, \quad \lim_{t\to-\infty} f_\infty(k,t) = -\infty, \quad k \in \mathbb{Z}[1,N],$

$(f_0^\pm)$ there exists $\delta > 0$ such that for $k \in \mathbb{Z}[1,N], \ |t| \leqslant \delta, \ \pm F_0(k,t) \geqslant 0.$

Our main results are the following theorems:

Theorem 1.1. *If $\nu_0 = \nu_\infty = 0$ and $\mu_0 \neq \mu_\infty$, then* (1.1) *has at least two nontrivial solutions.*

Theorem 1.2. *If $\nu_0 = 0$ and $\nu_\infty = 1$, then* (1.1) *has at least two nontrivial solutions in each of the following cases:*

(i) *(f_∞^-) and $\mu_0 \neq \mu_\infty$;*
(ii) *(f_∞^+) and $\mu_0 \neq \mu_\infty + 1$.*

Theorem 1.3. *If $\nu_0 = 1$ and $\nu_\infty = 0$, then* (1.1) *has at least two nontrivial solutions in each of the following cases:*

(i) *(f_0^-) and $\mu_0 \neq \mu_\infty$;*
(ii) *(f_0^+) and $\mu_\infty \neq \mu_0 + 1$.*

Theorem 1.4. *If $\nu_0 = \nu_\infty = 1$, then* (1.1) *has at least two nontrivial solutions in each of the following cases:*

(i) *(f_0^-), (f_∞^+) and $\mu_0 \neq \mu_\infty + 1$;*
(ii) *(f_0^+), (f_∞^-) and $\mu_\infty \neq \mu_0 + 1$.*

Due to the wide application in many fields such as computer science, economics, mechanical engineering, artificial or biological neural networks, the mathematical modelling of important questions, the theory of nonlinear discrete problem has been widely studied since the 1970s (see [1–4]).

Here, we are interested in investigating nonlinear discrete boundary value problems by using variational methods. The study of the solvability for discrete problem by this approach was initiated by Guo and Yu [5]. Since then, critical point theory has been extensively applied in showing the existence and the multiplicity of solutions for certain types of discrete problem (see [6–12]).

In recent years, many authors used critical point theory, mountain pass theorem, minmax methods and Mores theory to study the existence and the multiplicity of solutions for certain types of discrete problem when the nonlinearity is resonant (see [13–17]). The aim of this paper is to study the multiplicity of nontrivial solutions of problem (1.1) with asymptotically linear nonlinearity both at infinity and at zero which is the general of the resonant problem. Our technical approach is based on critical point theory and Morse theory.

2. Preliminaries

In this section we collect some results on Morse theory (see [18,19]). Let E be a Hilbert space and $J \in C^1(E, \mathbb{R})$. We say that J satisfies (PS) condition if every sequence $\{u_n\} \subset E$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.