Applied Mathematics and Computation 247 (2014) 494–500

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On Stirling's formula remainder

Long Lin

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454000, Henan Province, People's Republic of China

ARTICLE INFO

Keywords:
Gamma function
Psi function
Completely monotonic functions
Stirling's formula
Remainder in the Stirling formula
Estimates

ABSTRACT

Let the sequence r_n be defined by

$$n! = \sqrt{2\pi n}(n/e)^n e^{r_n}.$$

We establish new estimates for Stirling's formula remainder r_n. This improves some known results. Let $\theta(x)$ be defined by the relation:

$$\Gamma(x+1) = \sqrt{2\pi}(x/e)^x e^{\theta(x)/(12x)}.$$

We prove that $\theta''(x)$ is completely monotonic on $(0,\infty)$. This implies the result given by Mortici, who proved that $-x^{-1}\theta'''(x)$ is completely monotonic on $(0,\infty)$.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1,2,\ldots\} \tag{1}$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

In 1940 Hummel [9] defined the sequence r_n by

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{r_n}, \tag{2}$$

and established

$$\frac{11}{2} < r_n + \ln\sqrt{2\pi} < 1. \tag{3}$$

After the inequality (3) was published, many improvements have been given. For example, Robbins [18] established

E-mail address: linlong1978@sohu.com

http://dx.doi.org/10.1016/j.amc.2014.09.012
0096-3003/

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}. \tag{4}$$

Nanjundiah [17] proved

$$\frac{1}{12n} - \frac{1}{360n^3} < r_n < \frac{1}{12n}. \tag{5}$$

Beesack [4] found the following result:

$$\frac{1}{12n} - \frac{1}{360n^3} < r_n < \frac{1}{12n} - \frac{1}{\left(360 + \frac{30(7n(n+1)+1)}{n^2(n+1)^2}\right)n^3} \tag{6}$$

and recently Shi, Liu and Hu [19] provided the following result:

$$\frac{1}{12n} - \frac{1}{360n^3} < r_n < \frac{1}{12n} - \frac{1}{360n(n+1)(n+2)}. \tag{7}$$

Very recently, Chen and Batir [5] obtained the following sharp form of the inequality (4): For all integers $n \in \mathbb{N}$,

$$\frac{1}{12n+a} \leqslant r_n < \frac{1}{12n+b}$$

with the best possible constants

$$a = \frac{1}{1-\ln(\sqrt{2\pi})} - 12 = 0.336317\ldots \quad \text{and} \quad b = 0.$$

Our first aim in this work is to establish new estimates for r_n (see Theorem 1).

Euler's gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \qquad \mathrm{Re}(z) > 0$$

is one of the most important functions in mathematical analysis and other areas of mathematics. The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is known as the psi (or digamma) function. The derivatives of the psi function $\psi(z)$:

$$\psi^{(n)}(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

The gamma function $\Gamma(z+1)$ can be represented as follows for $\mathrm{Re}(z) > 0$,

$$\Gamma(z+1) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z e^{\mu(z)}.$$

It is well known that $\mu(x)$ is completely monotonic on $(0,\infty)$, since

$$\mu(x) = \frac{1}{2}\int_0^\infty \frac{Q(t)}{(x+t)^2}\,\mathrm{d}t,$$

where Q is periodic with period 1 and $Q(t) = t - t^2$ for $0 < t < 1$.

Recently, Shi, Liu and Hu [19] introduced the following generalized Stirling formula in a new approach (see [20, p. 253])

$$\Gamma(x+1) = \sqrt{2\pi}(x/e)^x e^{\theta(x)/(12x)} \tag{8}$$

and proved [19, Theorem 3] that $\theta(x)$ is a strictly increasing function for real numbers $x \geqslant 1$.

Very recently, Mortici [15] gave a complete characterization of the monotonicity of the function θ on its domain $(0,\infty)$. Precisely, Mortici proved that θ is strictly decreasing on $(0,\beta)$ and strictly increasing on (β,∞); where $\beta = 0.34142\ldots$ is the unique solution of the equation

$$\ln\Gamma(x+1) + x\psi(x+1) - 2x\ln x + x - \frac{1}{2}\ln(2\pi) = 0.$$

Moreover, the author showed that the function θ is strictly convex on $(0,\infty)$ and $-x^{-1}\theta'''(x)$ is completely monotonic on $(0,\infty)$. We recall that a function f is said to be completely monotonic on $(0,\infty)$ if it has derivatives of all orders on $(0,\infty)$ and satisfies the following inequality: