Applied Mathematics and Computation 247 (2014) 824–834

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# A generalized Conway–Maxwell–Poisson distribution which includes the negative binomial distribution

Tomoaki Imoto

*The Institute of Statistical Mathematics, 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan*

ARTICLE INFO

*Keywords:*
Excess zeros distribution
Index of dispersion
Log-concavity
Long-tailed
Queueing process
Statistical inference

ABSTRACT

The Conway–Maxwell–Poisson (COM-Poisson) distribution with two parameters was originally developed as a solution to handling queueing systems with state-dependent arrival or service rates. This distribution generalizes the Poisson distribution by adding a parameter to model over-dispersion and under-dispersion and includes the geometric distribution as a special case and the Bernoulli distribution as a limiting case. In this paper, we propose a generalized COM-Poisson (GCOM-Poisson) distribution with three parameters, which includes the negative binomial distribution as a special case, and can become a longer-tailed model than the COM-Poisson distribution. The new parameter plays the role of controlling length of tail. The GCOM-Poisson distribution can become a bimodal distribution where one of the modes is at zero and is applicable to count data with excess zeros. Estimation methods are also discussed for the GCOM-Poisson distribution.


## 1. Introduction

In statistical research, it is important to select an adequate distribution to describe the observed variation of counts. The Poisson distribution is a classically utilized model for analyzing count data. However, it has a serious restriction that the variance is equal to the mean or, equivalently, the index of dispersion (the ratio of variance to mean) is one, because observed count data do not satisfy the equality of the sample mean and variance in many cases. For many observed count data, it is common to have the sample variance to be greater or smaller than the sample mean which are referred to as over-dispersion and under-dispersion, respectively, relative to the Poisson distribution. Information on dispersion is useful for selecting an appropriate model to count data. For example, the negative binomial distribution is often selected for over-dispersed data and the binomial distribution is for under-dispersed data.

Shmueli et al. [7] have revived the Conway–Maxwell–Poisson (COM-Poisson for short) distribution, originally developed by Conway and Maxwell [2] as a solution to handling queueing systems with state-dependent arrival or service rates, and indicated its flexibility to adapt to over- and under-dispersions. The COM-Poisson distribution has the probability mass function (pmf)

$$P(x) = \frac{\lambda^x}{(x!)^r}\frac{1}{Z(\lambda,r)}, \quad \text{where} \quad Z(\lambda,r) = \sum_{k=0}^{\infty}\frac{\lambda^k}{(k!)^r} \tag{1}$$

for $r > 0$ and $\lambda > 0$ and reduces to the geometric distribution when $r \to 0$ and $0 < \lambda < 1$ and the Bernoulli distribution when $r \to \infty$. This means that the COM-Poisson distribution can become an over- or under-dispersed model. This flexibility greatly expands the types of problems for which the COM-Poisson distribution can be used to model count data.

http://dx.doi.org/10.1016/j.amc.2014.09.052
0096-3003/

In empirical modeling, the length of the tail parts of the distribution is an important factor. The negative binomial distribution is a generalized form of the geometric distribution and becomes a longer-tailed distribution. This paper proposes a generalization of the COM-Poisson (GCOM-Poisson for short) distribution, which includes the negative binomial distribution as a special case and, therefore, can become a longer-tailed model than the original COM-Poisson distribution. Moreover, the GCOM-Poisson can become a bimodal distribution where one of the modes is at zero and, therefore, can be adapted to count data with excess zeros. The flexibility of the dispersion and the length of the tail and applicability to excess zeros make the proposed distribution more versatile than the COM-Poisson distribution.

This paper is arranged as follows. The definition of the GCOM-Poisson distribution with some properties is given in Section 2. In Section 3, we consider methods of estimation for fitting the proposed distribution to real data sets and numerical examples using the methods are given in Section 4. Finally, our conclusion is given in Section 5.

## 2. A generalized COM-Poisson distribution

### 2.1. Definition

A random variable $X$ is said to have the GCOM-Poisson distribution with three parameters $r,\ \nu$ and $\theta$ if

$$\mathrm{P}(X=x)=\frac{\Gamma(\nu+x)^r\theta^x}{x!C(r,\nu,\theta)},\quad x=0,1,\ldots, \tag{2}$$

where the normalizing constant $C(r.\nu,\theta)$ is given by

$$C(r,\nu,\theta)=\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)^r\theta^k}{k!}, \tag{3}$$

for $r<1,\ \nu>0$ and $\theta>0$ or $r=1,\ \nu>0$ and $0<\theta<1$. The ratios of consecutive probabilities are formed as

$$\frac{\mathrm{P}(X=x)}{\mathrm{P}(X=x-1)}=\frac{\theta(\nu-1+x)^r}{x} \tag{4}$$

and it can be seen that $C(r,\nu,\theta)$ converges for $r<1$ or $r=1$ and $|\theta|<1$. Hence, the parameter space of the GCOM-Poisson distribution is $r<1,\ \nu>0$ and $\theta>0$ or $r=1,\ \nu>0$ and $0<\theta<1$. This distribution reduces to the COM-Poisson distribution with parameters $1-r$ and $\theta$ when $\nu\to1$ and, therefore, includes geometric and Bernoulli distributions. Moreover, the GCOM-Poisson distribution reduces to the negative binomial distribution when $r=1$.

The pmf (2) can be rewritten as

$$\mathrm{P}(X=x)=\frac{\exp\{r\log\Gamma(\nu+x)+x\log\theta-\log C(r,\nu,\theta)\}}{x!}.$$

Therefore, we can see that the GCOM-Poisson distribution with pmf (2) is a member of exponential family with natural parameters $(r,\log\theta)$ when $\nu$ is a known or nuisance parameter.

Following the definition of weighted distribution introduced by Rao [5], the GCOM-Poisson distribution corresponds to the weighted version of the Poisson distribution with the weight function $w(x)=\Gamma(\nu+x)^r$. The Corollary 4 in Castillo and Pérez-Casany [1] confirms that the weighted Poisson distribution with the weight function $w(x)=\exp\{rt(x)\}$, where $t(\cdot)$ is a convex function, is over-dispersed for $r>0$ and under-dispersed for $r<0$. Since $\log\Gamma(\nu+x)$ is a convex function, the GCOM-Poisson distribution is over-dispersed for $0<r<1$ and under-dispersed for $r<0$.

And, from the ratios of successive probabilities (4), the GCOM-Poisson distribution is seen to become a longer-tailed model than the COM-Poisson distribution when $\nu>1$ and $0<r<1$ and a shorter-tailed model when $\nu>1$ and $r<0$. Thus, the new parameter $\nu$ controls length of tail of the distribution.

### 2.2. Queueing process

Conway and Maxwell [2] considered queueing systems with state-dependent arrival or service rates, which lead to the COM-Poisson distribution. The GCOM-Poisson distribution is also generated as a queueing model with arrival rate $\lambda(\nu+x)^r$ and service rate $\mu x$, where $x$ is the size of the queue. This means that service rate is directly proportional to the state and arrival rate increases as queue becoming large for $r>0$ and decreases as queue becoming large for $r<0$. Then $P(x,t)$, the probability of the size of the queue being $x$ at time $t$, satisfies the difference equation for small $h$

$$P(x,t+h)=\{1-\lambda(\nu+x)^rh-\mu xh\}P(x,t)+\lambda(\nu+x-1)^rhP(x-1,t)+\mu(x+1)hP(x+1,t).$$

Putting $\theta=\lambda/\mu$ and letting $h\to0$, we get the following difference–differential equation

$$\frac{\partial P(x,t)}{\partial t}=(\nu+x-1)^r\theta P(x-1,t)+\{(\nu+x)^r\theta+x\}P(x,t)+(x+1)P(x+1,t).$$

Assuming a stationary state and then letting $P(x,t)=P(x)$, we finally have the difference equation