Applied Mathematics and Computation 247 (2014) 865–871

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Synchronous sequences and inequalities for convex functions

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ARTICLE INFO

Keywords:
Chebyshev type inequality
Majorization type inequality
Synchronous sequences
Convex function

ABSTRACT

In this paper, we derive discrete inequalities of the majorization type for convex functions by using Chebyshev type inequalities for synchronous sequences (cf. Otachel, 2011). Thus, some of the related results from Dragomir (2004) and Niezgoda (2008) are unified and extended. Applications for particular convex functions are given.

1. Introduction and motivation

The classic Chebyshev inequality (see e.g. [14, sec. 7.1])

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \leqslant n \sum_{i=1}^{n} x_i y_i,$$

holds if sequences (x_i) and (y_i) are monotonic in the same direction or, more generally, are similarly ordered, or if they are monotonic in mean. In [9] is shown that the inequality holds if there exists a value which separates y_i in the same way as the arithmetic mean of x_i separates x_i. This is an example of similarly separable sequences (see [11]). Such type of sequences in a natural way generalizes monotonic sequences. The applicability of separable sequences for Chebyshev type inequalities can be simplified and extended beyond finite-dimensional spaces by introducing synchronous vectors w.r.t. dual bases in Banach spaces (see [13]). Two vectors are synchronous if their corresponding coordinates (w.r.t. given dual bases) are of the same sign. Thus one can tell about synchronous sequences in place of synchronous vectors. Using this notion we present a version of Chebyshev's inequality for infinite sequences and consequently, we obtain new assumptions implying inequalities of the form

$$\sum_{v} p_v f(y_v) \leqslant \sum_{v} p_v f(x_v), \tag{1}$$

where f is a real convex function, $\overline{x} = (x_v), \overline{y} = (y_v)$ are real sequences and $\overline{p} = (p_v)$ is a positive sequence.

Inequality (1), our main interest, is closely related to the theory of majorization (see [6]). If $\overline{p} \equiv 1$ and $\overline{p}, \overline{x}, \overline{y}$ are n-tuples, then (1) holds for all continuous convex functions f if and only if $\overline{x}$ majorizes $\overline{y}$ in the sense that the sum of k largest entries of $\overline{y}$ is not greater than the sum of k largest entries of $\overline{x}$ for all $k = 1, 2, \ldots, n$ with equality for $k = n$. This classic result is due to Hardy et al. [4]. References to other classical extensions one can find in [14, chap. 12]. Recent contributions come from [8,7,5,12] and others. In particular, Dragomir [3] obtained new sufficient conditions for (1) to hold by using Chebyshev's inequality for n-tuples of the same monotonicity and for vectors monotonic in $\overline{p}$-mean, respectively. Applications of Chebyshev's inequalities to inequality (1) were continued in [10,1]. Barnett et al. [1] gave integral versions of the mentioned results. Niezgoda [10] generalized Dragomir's results with the help of Chebyshev type inequalities for separable finite sequences.

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http://dx.doi.org/10.1016/j.amc.2014.09.037
0096-3003/

In this paper we unify and extend some of the results by Dragomir and Niezgoda. A key tool for this purpose is Chebyshev type inequality for synchronous sequences presented in Proposition 1, Section 2. It is a specification of [13, Proposition 2] for l^∞. Moreover, in Section 2 we also collect some essential properties of synchronous and convex sequences utilized in the sequel. General sufficient conditions for (1) are contained in Theorem 1, Section 3. The assumptions of this theorem depend on choice of dual bases in l^p-spaces. Next results of the section are corollaries from Theorem 1 for specific bases. For instance, employing canonical bases, we get in Corollary 1 generalizations of Dragomir's results [3, Corollary 1–2] on (1) for n-tuples of the same monotonicity. Simultaneously, Corollary 1 extends Niezgoda's result [10, Corollary 2.3] from $\mathbb{R}^n$ to l^∞. Corollary 2 provides assumptions for (1) to hold within the framework of relative convexity for infinite sequences. On the other hand, it corresponds with Chebyshev's inequality by Otachel [13, Corollary 5]. Corollary 3 concerns (1) for infinite sequences monotonic in mean. In particular situations, it reduces to the result by Dragomir (cf. [3, Corollary 4]) and extends Niezgoda's result [10, Corollary 2.6] to l^∞. Applications for particular convex functions are contained in Section 4. We obtain there further variants and refinements of (1).

2. Synchronous sequences and Chebyshev's inequality

Throughout the paper, all the sequences are real and infinite. For every sequence $\bar{x}$ by x_i we denote its ith entry. Sometimes, a sequence with ith entry equal to x_i will be written as (x_i). For sequences $\bar{x},\bar{y}$ and a real function ϕ with a properly chosen domain by $\phi\circ\bar{x},\bar{x}\cdot\bar{y}$ and $\bar{x}/\bar{y}$ we denote the sequences $(\phi(x_i)),(x_iy_i)$ and (x_i/y_i), provided that $y_i\neq 0$, respectively. The simplified notation for the operation of summing will be used, e.g. $\sum\bar{x},\sum\bar{x}\cdot\bar{y}$ stand for $\sum_i x_i,\sum_i x_iy_i$, etc.

Two sequences $\bar{a}=(a_i)$ and $\bar{b}=(b_i)$ are said to be: *synchronous* (abbrev. $\bar{a}\sim\bar{b}$) if $a_ib_i\geq 0$ for all integer i; *similarly ordered* if $[a_i-a_j][b_i-b_j]\geq 0$ for any i,j.

It is evident, that (a_i) and (b_i) are similarly ordered if and only if the sequences $(a_i-\lambda)$ and $(b_i-\gamma)$ are similarly ordered for arbitrary scalars λ and γ. Other relevant properties of such sequences are included below.

Lemma 1.

(i) *If (a_i) and (b_i) are similarly ordered and (b_i) is bounded, then there exists a scalar λ such that $(a_i)\sim(b_i-\lambda)$.*
(ii) *$(a_i)\sim(b_i)$ if and only if there exist complementary subsets of integers σ and ϱ such that*

$$a_j,b_j\leq 0\leqslant a_i,b_i,\quad i\in\sigma,\quad j\in\varrho.$$

(iii) *Given three sequences $(a_i),(b_i)$ and (c_i), where (b_i) and (c_i) are similarly ordered, $b_i=b_j$ implies $c_i=c_j$ for integers i,j and (c_i) is bounded.*
(iv) *If $(a_i)\sim(b_i)$, then there exists a scalar λ such that $(a_i)\sim(c_i-\lambda)$.*
(v) *If $(a_i)\sim(b_i)$ and $(b_i)\sim(c_i)$, then $(a_i)\sim(c_i)$ whenever $b_i=0$ implies $c_i=0$ for all i.*

Proof.

(i) Set $\sigma=\{i:0\leqslant a_i\}$ and let ϱ be the complementary sequence with σ. If $\sigma\neq\emptyset$, then we define $\lambda=\inf\{b_i:i\in\sigma\}$ else $\lambda=\sup_ib_i$. Clearly, $\lambda\leqslant b_i,\ i\in\sigma$. We shall show that $b_j\leqslant\lambda,\ j\in\varrho$. It is nothing but $(a_i)\sim(b_i-\lambda)$, because $a_j<0\leqslant a_i,\ i\in\sigma,j\in\varrho$.
Suppose that there exists $j\in\varrho$ with $\lambda<b_j$. It forces existing of $i\in\sigma$ such that $\lambda<b_i<b_j$, so $b_i-b_j<0$ for certain $i\in\sigma,\ j\in\varrho$. On the other hand, $a_j<0\leqslant a_i$. Hence $[a_i-a_j][b_i-b_j]\leq 0$, a contradiction.
(ii) Set $\sigma=\{i:a_i>0\}\cup\{i:a_i=0,\ b_i\geq 0\}$ and let ϱ be the complementary sequence with σ, i.e. $\varrho=\{i:a_i<0\}\cup\{i:a_i\leq 0,\ b_i<0\}$. Clearly, $a_j\leq 0\leqslant a_i,\ i\in\sigma,\ j\in\varrho$.
If $(a_i)\sim(b_i)$, then $a_i>0$ implies $b_i\geq 0$ and $a_i<0$ implies $b_i\leq 0$. Therefore $b_j\leqslant 0\leqslant b_i,\ i\in\sigma,\ j\in\varrho$.
The converse implication is evident.
(iii) By (ii), if $(a_i)\sim(b_i)$, then there exist complementary subsets of integers σ and ϱ such that $a_j,b_j\leq 0\leqslant a_i,b_i\ i\in\sigma,\ j\in\varrho$. Particularly, $b_j<0<b_i$ for $i\in\sigma,j\in\varrho$ with $b_i\neq 0\neq b_j$. Hence $c_j\leqslant c_i$, because (b_i) and (c_i) are similarly ordered. If $b_i=0=b_j,\ i\in\sigma,j\in\varrho$, then $c_i=c_j$, by the hypothesis. In a consequence, $c_j\leqslant c_i$ for all $i\in\sigma$ and $j\in\varrho$. If $\sigma\neq\emptyset$, then we define $\lambda=\inf\{c_i:i\in\sigma\}$ else $\lambda=\sup_ic_i$. It is easily seen that $c_j\leqslant\lambda\leqslant c_i,\ i\in\sigma,\ j\in\varrho$. Finally, we have $a_i\geq 0,\ c_i\geqslant\lambda,\ i\in\sigma$ and $a_j\leq 0,\ c_j\leqslant\lambda,\ j\in\varrho$, or equivalently, $a_k[c_k-\lambda]\geq 0$ for all k.
(iv) For all integer k, $a_kb_k\geq 0,\ b_kc_k\geq 0$ implies $a_kb_k^2c_k\geq 0$. If $b_k\neq 0$, then $a_kc_k\geq 0$. If $b_k=0$, then $c_k=0$ and $a_kc_k=0$. Therefore, $a_kc_k\geq 0$ for all k. □

A sequence (x_i) is said to be a *convex sequence* (cf. [14, Def. 1.11]) if $x_{i+1}-x_i\leqslant x_{i+2}-x_{i+1}$, for all i. A more general notion is a concept of relative convexity. Let $\overline{w}$ be a nonconstant sequence. We say that a sequence $\bar{x}$ is *convex with respect to* $\overline{w}$ (abbrev. $\overline{w}\triangleleft\bar{x}$) if $\begin{vmatrix}1 & w_i & x_i\\ 1 & w_j & x_j\\ 1 & w_k & x_k\end{vmatrix}\geq 0$ whenever $w_i\leqslant w_j\leqslant w_k$, (for the general definition see e.g. [8, Definition 1]). It is clear that (x_i) is convex if and only if $(i)\triangleleft(x_i)$.