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New algorithms for Taylor coefficients of indefinite integrals and their applications

Shih-Hsiang Chang

Department of Mechanical Engineering, Far East University, No. 49, Jhonghua Rd., Sinshih, Tainan 74448, Taiwan, ROC

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ABSTRACT

New recurrence algorithms for calculating the Taylor coefficients of indefinite integral are introduced and proved in this paper. These algorithms are simple in form, only involve the analytic operations: addition and multiplication, and can be directly implemented in any symbolic language. Also, no specific restriction on the kernel is required. Several types of Volterra equations such as integral, integro-differential and system of integro-differential equations with nondegenerate kernel are then solved to demonstrate the accuracy and efficiency of the proposed scheme.

1. Introduction

A detailed comparison between Taylor series method (TSM) and differential transform method (DTM) has been recently carried out by Bervillier [1] and the study showed that DTM exactly coincides with TSM when DTM is applied to ordinary differential equations. TSM has been successfully applied to a wide class of nonlinear differential equations without requiring linearization, discretization or perturbation [1–8]. The fundamental correspondence between one- or two-variable functions and their Taylor coefficients as shown in Tables 1 and 2 were already known and used for a long time by the TSM user [9–12]. Other iterative formulas for calculating the Taylor coefficients of nonlinear functions such as exponential, logarithm and trigonometric nonlinearities have been well established [13–18]. For further details, see [1] and references therein.

Further application of this method to the Volterra type equations relies on the development of efficient algorithms for calculating the Taylor coefficients of indefinite integrals. An iterative formula for single indefinite integral with general kernel has been derived by Sezer [19]. Recently, similar formulas have been extended to the multiple indefinite integrals [20,21]. However, the kernel $K(x,s,u(s))$ used in these two studies is restricted to the degenerate type, i.e.,

$$K(x,s,u(s)) = \sum_{i=0}^{n} k_i(x) g_i(s,u(s)) \tag{1}$$

where $u(s)$ is the unknown function to be determined.

In this paper, the algorithm proposed in previous work [19] has been generalized and extended to the multiple indefinite integrals. The present algorithms offer a computationally easier approach to computing the Taylor coefficients for all form of indefinite integral. No specific restriction on the kernel is required. Also, these algorithms consist of simple, orderly and analytic recurrence formulas, and are very easy to implement by using any symbolic language such as *MATHEMATICA* or

E-mail address: shchang@cc.fec.edu.tw

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Table 1
The fundamental correspondence between one-variable functions and their Taylor coefficients.

original function	Taylor coefficient
$w(x) = \alpha u(x) \pm \beta v(x)$	$W(k) = \alpha Y(k) \pm \beta Z(k)$
$w(x) = \frac{d^n u(x)}{dx^n}$	$W(k) = \frac{(k+n)!}{k!} U(k+n)$
$w(x) = u(x)v(x)$	$W(k) = \sum_{m=0}^{k} U(m)V(k-m)$
$w(x) = x^n$	$W(k) = \delta(k-n)$

Table 2
The fundamental correspondence between two-variable functions and their Taylor coefficients.

original function	Taylor coefficient
$w(x,y) = \alpha u(x,y) \pm \beta v(x,y)$	$W(k,h) = \alpha Y(k,h) \pm \beta Z(k,h)$
$w(x,y) = \frac{\partial^{m+n} u(x,y)}{\partial x^m \partial y^n}$	$W(k,h) = \frac{(k+m)!(h+n)!}{k!h!} U(k+m, h+n)$
$w(x,y) = u(x,y)v(x,y)$	$W(k,h) = \sum_{m=0}^{k}\sum_{n=0}^{h} U(m, h-n)V(k-m, n)$
$w(x,y) = x^m y^n$	$W(k,h) = \delta(k-m)\delta(h-n)$

MAPLE. The accuracy and efficiency of the algorithms are assessed and tested on several Volterra-type equations with different kind of nondegenerate kernels.

2. Formulation of algorithms

As mentioned by Bervillier [1], DTM provides a convenient way to analytically obtain the Taylor coefficients via an iterative equation; hence, the notation used in DTM is adopted in this paper.

If the function $u(x)$ is analytic in an interval with center at $x = a$, then its Taylor series expansion about this center is

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(x)}{dx^k}\bigg|_{x=a} (x-a)^k \tag{2}$$

and the corresponding Taylor coefficient $U(k)$ is defined as

$$U(k) = \frac{1}{k!} \frac{d^k u(x)}{dx^k}\bigg|_{x=a} \tag{3}$$

Similarly, if the function $u(x,y)$ is analytic in a region with center at $(x,y) = (a,b)$, then its Taylor series expansion of $u(x,y)$ about this center is

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!} \frac{\partial^{k+h} u(x,y)}{\partial x^k \partial y^h}\bigg|_{\substack{x=a\\y=b}} (x-a)^k (y-b)^h \tag{4}$$

and the corresponding Taylor coefficient $U(k,h)$ is

$$U(k,h) = \frac{1}{k!h!} \frac{\partial^{k+h} u(x,y)}{\partial x^k \partial y^h}\bigg|_{\substack{x=a\\y=b}} \tag{5}$$

In the following theorems, all of the functions in the kernel are assumed to be analytic at a point $x = a$ or $(x,y) = (a,a)$ with a positive radius of convergence.

Theorem 1. *If* $w(x) = \int_a^x u(x,y)dy$, *then* $W(0) = 0$ *and* $W(k) = \sum_{n=0}^{k-1} \frac{U(k-n-1,n)}{n+1}, \quad k \geqslant 1.$

Proof. By the definition of Taylor coefficient, $W(0) = \frac{1}{0!} w(x=a) = 0$. Since the function $u(x,y)$ is analytic at a point $(x,y) = (a,a)$, then its Taylor series expansion about this point can be expressed as

$$u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} U(m,n)(x-a)^m (y-a)^n$$

from which it follows