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Bi-seasonal discrete time risk model ☆

Julius Damarackas, Jonas Šiaulys *

Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania

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ABSTRACT

This work is concerned with a bi-seasonal discrete time risk model for insurance. Specifically, the claims repeat with time periods of two units, i.e. claim distributions coincide at all even instants and at all odd instants. Our purpose is to derive recursive formulas to calculate the finite-time and ultimate ruin probabilities. Some numerical examples illustrate the theoretical results.

1. Introduction

The discrete time risk model is a classical collective model for insurance. In the homogeneous version of this model, the insurer's surplus at each time moment $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is described by the following equation:

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i, \tag{1.1}$$

where $u \in \mathbb{N}_0$ is the initial insurer's surplus. Claim amounts $Z_1, Z_2, \ldots$ are assumed to be independent copies of a nonnegative integer-valued random variable Z. It is evident that this random variable and the initial surplus u generate the discrete time risk model. The claim amount generator Z is of probability mass function (p.m.f.)

$$\mathbb{P}(Z = k) = z_k, \quad k = 0, 1, \ldots,$$

or by the distribution function (d.f.)

$$F_Z(x) = \sum_{k=0}^{[x]} z_k, \; x \in \mathbb{R},$$

where $[x]$ is the integer part of x.

This model has been extensively investigated by Dickson ([9,10]), De Vylder and Goovaerts ([7,8]), Gerber [12], Seal [24], Shiu ([25,26]), Picard and Lefèvre ([20–22]), Lefèvre and Loisel [15], Leipus and Šiaulys [17], Tang [27] and other authors.

The time of ruin, the ruin probability and the finite time ruin probability are the main extremal characteristics of discrete time risk model in (1.1). The first time T_u when the surplus W_u becomes negative or null is called *the time of ruin*, i.e.

$$T_u = \begin{cases} \inf\{n \geqslant 1 : \; W_u(n) \leqslant 0\}, \\ \infty \;\text{ if }\; W_u(n) > 0 \;\text{ for all }\; n \in \mathbb{N}. \end{cases}$$

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* Corresponding author.
E-mail addresses: julius.damarackas@gmail.com (J. Damarackas), jonas.siaulys@mif.vu.lt (J. Šiaulys).

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The probability of ruin until time moment $T \in \mathbb{N}$ is called *the finite time ruin probability* and is defined by equality

$$\psi(u,T) = \mathbb{P}(T_u \leqslant T).$$

The infinite time or *ultimate ruin probability* is defined by equality

$$\psi(u) = \mathbb{P}(T_u < \infty).$$

Clearly,

$$\psi(u,T) = \mathbb{P}\left(\bigcup_{n=1}^{T}\left\{u+n-\sum_{i=1}^{n} Z_i \leqslant 0\right\}\right) = \mathbb{P}\left(\max_{1\leqslant n\leqslant T}\sum_{i=1}^{n}(Z_i-1) \geqslant u\right),$$

$$\psi(u) = \mathbb{P}\left(\sup_{n\geqslant 1}\sum_{i=1}^{n}(Z_i-1) \geqslant u\right),$$

$$\psi(u,T) \nearrow \psi(u), \quad T\to\infty. \tag{1.2}$$

Several formulas and procedures for computing finite time ruin probability have been proposed in the literature. Here we present some of them.

- The Seal's formula (see [24]):

$$\psi(u,T) = 1 - \sum_{j=0}^{u+T-1} z_j^{*T} + \sum_{j=u+1}^{u+T-1} z_j^{*(j-u)}\left(\sum_{n=j}^{u+T-1}\frac{T+u-n}{T+u-j}\, z_{n-j}^{*(T+u-j)}\right), \quad T=1,2,\ldots.$$

  Here, $z_j^{*m}, j\in\mathbb{N}_0, m\in\mathbb{N} := \{1,2,\ldots\}$, denote the local probabilities: $z_j^{*m} = \mathbb{P}(Z_1+\ldots+Z_m = j)$, where $Z_1,\ldots,Z_m$ are independent copies of random claim amount Z.

- The Picard-Lefèvre formula (see [20]):

$$\psi(u,T) = 1 - \sum_{j=0}^{u}\left(z_j^{*T} + \hat{z}_j^{*(j-u)}\sum_{n=u+1}^{u+T-1}\frac{T+u-n}{T+u-j}\, z_{n-j}^{*(T+u-j)}\right), \quad T=1,2,\ldots.$$

  Here, $z_j^{*m}, j\in\mathbb{N}_0, m\in\mathbb{N}$, are the same as in Seal's formula, and

$$\hat{z}_j^{*t} = z_0^t\sum_{k=1}^{j}\frac{t(t-1)\ldots(t-k+1)}{k!}\left(\frac{1-z_0}{z_0}\right)^k \mathbb{P}(\hat{S}_k = j), \quad j\in\mathbb{N}_0, \quad t\in\mathbb{Z},$$

  where $\hat{S}_0 = 0,\ \hat{S}_k = \hat{Z}_1+\ldots+\hat{Z}_k, k\in\mathbb{N}$, with independent identically distributed (i.i.d.) random variables $\hat{Z}_1, \hat{Z}_2, \ldots$, distributed according to d.f. $\mathbb{P}(Z\leqslant x | Z\geqslant 1)$.

  The Picard–Lefèvre formula has many generalizations and extensions which can be applied in various insurance risk models (see [5,6,23]).

- The recursive formula (see, for instance, [7,11,10]):

$$\psi(u,1) = 1 - F_Z(u),$$

$$\psi(u,T) = \psi(u,1) + \sum_{k=0}^{u}\psi(u+1-k, T-1) z_k, \quad T=2,3,\ldots.$$

Obtaining $\psi(u,T)$ for large T, due to formula (1.2) we get the lower bound for the ultimate ruin probability. If there exists a suitable estimate of $\psi(u,T)$ for all large T, using the same formula the upper bound for $\psi(u)$ can be obtained.

On the other hand, there is a recursive procedure to calculate the values of the infinite time ruin probability. Namely, under assumption $\mathbb{E}Z < 1$ we have that:

$$\begin{cases}\psi(0) = \mathbb{E}Z, \\ \psi(u) = \sum_{j=1}^{u-1}(1-F_Z(j))\psi(u-j) + \sum_{j=u}^{\infty}(1-F_Z(j)), \quad u=1,2,\ldots\end{cases} \tag{1.3}$$

according to [10] (see also [11,25]). If $\mathbb{E}Z \geqslant 1$, then the model does not satisfy the net profit condition and, in such a case, we have that $\psi(u) = 1$ for all $u\in\mathbb{N}$ according to the general theory on the renewal risk models (see, for instance, [19] and references therein).

There exist many other methods which allow to calculate or estimate the ultimate ruin probability. Some of them can be found in [1,18,21,28].