Applied Mathematics and Computation 247 (2014) 1052–1063

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Subgradient algorithms for solving variable inequalities ☆

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ARTICLE INFO

Keywords:
Convexity
Projection methods
Subgradient methods
Variable ordering

ABSTRACT

In this paper we consider the variable inequalities problem, that is, to find a solution of the inclusion given by the sum of a function and a point-to-cone application. This problem can be seen as a generalization of the classical inequalities problem taking a variable order structure. Exploiting this relation, we propose two variants of the subgradient algorithm for solving the variable inequalities model. The convergence analysis is given under convex-like conditions, which, when the point-to-cone application is constant, contains the old subgradient schemes.

1. Introduction

We consider the inclusion problem of finding $x \in C$ such that

$$0 \in T(x), \tag{1}$$

where $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a point-to-set operator and C is a nonempty and closed subset of $\mathbb{R}^n$. Inclusions has been studied in many works due its applications; see, for instance, [30,14,28]. However, we will focus in the case in which $T(x) = F(x) + K(F(x))$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ and $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a point-to-set application such that $K(y)$ is a closed pointed convex cone for all $y \in \mathbb{R}^m$. Then, we are lead to the model:

$$\text{find a point } x \in C \text{ fulfilling that } 0 \in F(x) + K(F(x)). \tag{2}$$

If K is a constant application, problem (2) is equivalent to compute $x \in C$ such that

$$0 \in F(x) + K. \tag{3}$$

This model is known as the K-inequalities problem because, using the partial order defined in $\mathbb{R}^m$ by K as

$$\hat{y} \preceq_K y \quad \text{if and only if} \quad y - \hat{y} \in K,$$

problem (3) is equivalent to:

$$\text{find } x \in C \text{ such that } F(x) \preceq_K 0. \tag{4}$$

Model (2) can be interpreted as a system of variable inequalities. Indeed, consider the variable order given by

☆ This work was partially supported by CNPq – Brazil Grants 303492/2013-9, 474160/2013-0 and 202677/2013-3 and by project CAPES-MES-CUBA 226/2012.

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http://dx.doi.org/10.1016/j.amc.2014.09.073

$$z \preceq_{K(z)} y \quad \text{if and only if} \quad y - z \in K(z);$$

see [17,16] for more details. Then, problem (2) is equivalent to:

$$\text{find } x \in C \text{ such that } F(x) \preceq_{K(F(x))} 0. \tag{5}$$

That is why, from now on, this problem will be called the variable inequalities problem. The solution set of this problem will be denoted by S_*.

Note that if K is a constant application, problem (5) leads to model (4), which has been already studied in [26,27,10,11]. Moreover, if K is the Pareto cone, *i.e.*, $K = \mathbb{R}^m_+$, it is equivalent to the convex feasibility problem, which has been well-studied in [4] and has many applications in optimization theory, approximation theory, image reconstruction and so on; see, for instance, [25,31,13]. The variable case is not only a generalization of problem (4). Variable order optimization models appear in portfolio and medicine applications, as recently reported in [2,3,16].

The algorithms for solving problem (4) mainly converge under convexity of F. We generalize this concept to the variable order case as follows

$$\alpha F(x) + (1-\alpha)F(\hat{x}) - F(\alpha x + (1-\alpha)\hat{x}) \in K(F(\alpha x + (1-\alpha)\hat{x})). \tag{6}$$

We want to point out that relation (6) generalizes the previously defined convexity concept to the case in which the point-to-cone application, K, is identically constant. As in this case, if F is a K-convex function and C is a convex set, model (5) is also called a K-convex inequalities problem.

In this paper we propose a subgradient approach for solving problem (5), which combines a subgradient iteration with a simple projection step, onto the intersection of C with suitable halfspaces containing the solution set S_*. The proposed conceptual algorithm has two variants called Algorithms R and S. The first one is based on Robinson's subgradient algorithm given in [27] for solving problem (4). The S variant corresponds to a special modification of the subgradient algorithms proposed in [9] for the scalar problem ($m = 1$ and $K = \mathbb{R}_+$) and in [10] for solving problem (4). The main difference between the proposed variants lies in how the projection step is done. For the convergence of the variants, we assume that the set S_* is nonempty and that the function F is K-convex with respect to the defined variable order extending the previous schemes.

The paper is organized as follows. In the next section, we outline the main definitions and preliminary results. In Section 3 some analytical results and comparisons for K-convex functions are established. Section 4 is devoted to the presentation of the algorithms and their convergence is shown in Section 5. Finally, some comments and remarks are presented in Section 6.

2. Preliminaries

In this section, we present some definitions and results, which are needed in the convergence analysis. We begin with some classical notations.

The inner product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, the norm, induced by this inner product, by $\| \cdot \|$ and $B[x, \rho]$ is the closed ball centered at $x \in \mathbb{R}^n$ with radio ρ, *i.e.*, $B[x, \rho] := \{y \in \mathbb{R}^n : \|y - x\| \leqslant \rho\}$. A set valued application $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is closed if and only if $gr(K) := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : y \in K(x)\}$ is a closed set. Given the cone $\mathcal{K}$, the dual cone of $\mathcal{K}$, denoted $\mathcal{K}^*$, is $\mathcal{K}^* := \{z \in \mathbb{R}^m : \langle z, y \rangle \geqslant 0, \ \forall y \in \mathcal{K}\}$.

The set C will be a closed and convex subset of $\mathbb{R}^n$. For an element $x \in \mathbb{R}^n$, we define the orthogonal projection of x onto $C, P_C(x)$, as the unique point in C, such that $\|P_C(x) - y\| \leqslant \|x - y\|$ for all $y \in C$. In the following we consider a well known fact on orthogonal projections.

Proposition 2.1. *Let C be a nonempty, closed and convex set in $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$ and all $z \in C$, the following property holds: $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$.*

Proof. See Theorem 3.14 of [5]. □

Next we deal with the so-called Fejér convergence and its properties.

Definition 2.1. Let S be a nonempty subset of $\mathbb{R}^n$. A sequence $(x^k)_{k\in\mathbb{N}}$ is said to be Fejér convergent to S, if and only if for all $x \in S$, there exists $\bar{k} > 0$ such that $\|x^{k+1} - x\| \leqslant \|x^k - x\|$ for all $k \geqslant \bar{k}$.

This definition was introduced in [12] and has been further elaborated in [20]. An useful result on Fejér sequences is the following.

Theorem 2.2. *If $(x^k)_{k\in\mathbb{N}}$ is Fejér convergent to S then,*

(i) The sequence $(x^k)_{k\in\mathbb{N}}$ is bounded,
(ii) if a cluster point of the sequence $(x^k)_{k\in\mathbb{N}}$ belongs to S, then the sequence $(x^k)_{k\in\mathbb{N}}$ converges to a point in S.

Proof. See Theorem 2.16 of [4]. □