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Proper orthogonal decomposition with high number of linear constraints for aerodynamical shape optimization

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ABSTRACT

Shape optimization involving finite element analysis in engineering design is frequently hindered by the prohibitive cost of function evaluations. Reduced-order models based on proper orthogonal decomposition (POD) constitute an economical alternative. However the truncation of the POD basis implies an error in the calculation of the global values used as objectives and constraints which in turn affects the optimization results. In our former contribution (Xiao and Breitkopf, 2013), we have introduced a constrained POD projector allowing for exact linear constraint verification for a reduced order model. Nevertheless, this approach was limited a to relatively low numbers constraints. Therefore, in the present paper, we propose an approach for a high number of constraints. The main idea is to extend the snapshot POD by introducing a new constrained projector in order to reduce both the physical field and the constraint space. This allows us to search for the Pareto set of best compromises between the projection and the constraint verification errors thereby enabling fine-tuning of the reduced model for a particular purpose. We illustrate the proposed approach with the reduced order model of the flow around an airfoil parameterized with shape variables.

1. Introduction

Design optimization and uncertainty quantification of physical systems require frequent calls to the simulation models. The high CPU costs of individual simulations performed often on slightly varying configurations foster the development of reduced order modeling techniques.

The reduced order modeling domain covers a broad range of approaches which may be categorized either according to the degree of abstraction underlying physics or from the point of view of intrusiveness with respect to the original high fidelity numerical model. According to increasing complexity, the models range from simplified physics [12] and response surface modeling (RSM) [5] up to full field proper orthogonal decomposition (POD) [1] and high dimensional parametric representations based on proper orthogonal decomposition (PGD) [4]. In an *a posteriori* approach [19] proposes a combination of the proper orthogonal decomposition and concepts from balanced realization theory to obtain a balanced reduced-order model. The hyperreduction [16] is an *a priori* approach in which the ROM and the state evolution are simultaneously improved, by an adaptive strategy. A reduced order modeling approach based on dynamic flexibility approximation was

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applied to transient analyzes [14] and a flexibility-based component mode synthesis was proposed for reduced-order modeling of dynamic behavior of large structures [13]. The use of POD for problems governed by systems of nonlinear differential equations and in non-linear regimes has been treated by [17,2,3]. More theoretical details on the Galerkin projection are presented in [15] and in the scope of aeroelastic problems by [18].

In the present work we focus on an *a posteriori* bi-level approach, combining POD with RSM. This has been used in the scope of an optimization framework of coupled problems [8], in metal forming process [10], aerodynamic design [7] and flow optimization [20]. [9] combines RSM with PGD for optimization of the pultrusion process. The multi-level simulations involving various levels of model reduction are proposed by [11] for uncertainty quantification in stamping process. However, while the POD provides the estimator of the reduced field error, an additional work has to be done in order to control the quality of the integral quantities such as multiple objective and constraint functions intervening in the optimization process.

Our goal is to use the reduced order model within the shape optimization of a wing profile, in order to reduce the computational cost. We apply a simple model with flow function ψ expressed in terms of a set of basis vectors (modes) $\gamma_1, \ldots, \gamma_M$ and approximated by a subset of $m \ll M$ of modes $\tilde{\psi} = [\gamma_1, \ldots, \gamma_m]\psi$. The quality of the reduced order basis and its ability to represent the quantities of interest is of primary importance. The quantities of interest (lift *Cl* and pitch *Cp*) are computed by a linear operator $\mathbf{G}$ such that $C = (C_L, C_p)^T = \mathbf{G}\psi$. However, the truncation of the number of modes results in an error in the flow and the quantities of interest computed from the approximate flow $\tilde{\psi}$ are different from the values obtained without truncation $\mathbf{G}^T\psi \neq \mathbf{G}^T\tilde{\psi}$ yielding an error $\varepsilon_{\mathbf{C}} = ||\mathbf{G}^T[\gamma_{m+1}\gamma_{m+1}^T\psi + \cdots + \gamma_M\gamma_M^T\psi]||$.

To solve this problem, we define a reduced basis consistent with the flow and with the quantities of interest. Our idea is to adapt the first m basis vectors in the way that $M - m$ skipped modes $\gamma_{m+1}, \ldots, \gamma_M$ do not contribute to the product $\mathbf{G}\psi$. We construct the modes in a way such that $\gamma_{m+1}, \ldots, \gamma_M$ are in the nullspace of $\mathbf{G}^T$. Then, we obtain an exact reconstruction of (C_L, C_p) with $m < M$ as the error vanishes $\varepsilon_{C_L} = \mathbf{0}$.

In the former papers [21,22], we have introduced a constrained POD projector allowing for exact linear constraint verification for a reduced order model. However, this approach was limited to relatively low numbers of constraints such that rank $(G) < M$. In the present work, we propose an approach for a high number of constraints.

In the example section of the present paper, we treat a case with twice as many constraints as snapshots: for every snapshot we have two constraints (one on *Cm* and one on *Cl*) and we cannot expect rank $(G) < M$. So, instead of minimizing $||\psi - \tilde{\psi}||$ under the condition $\mathbf{G}^T\psi = \mathbf{G}^T\tilde{\psi}$, we solve a bi-objective minimization problem for two objectives: $||\psi - \tilde{\psi}||$ and $||\mathbf{G}^T\psi - \mathbf{G}^T\tilde{\psi}||$. We define two sets of modes Υ_1 and Υ_2 in order to control respectively the error of the quantities of interest and of the flow. As the two sets are orthogonal, both errors are controlled independently.

The paper is organized as follows. In the following Section 2 we recall the formulation of the POD projector as a minimization problem, both in its unconstrained and constrained versions. In the third section, we define the highly constrained projector in terms of the bi-objective minimization problem statement and we propose an algorithm for minimizing both objective functions. In Section 4, we illustrate the formulation on a problem of the flow around an airfoil parameterized with shape variables. The numerical results are discussed in Section 5, which allows us to draw conclusions and prospects in Section 6.

2. POD formulation and constrained case

In [22], we have revisited POD (proper orthogonal decomposition) [1] by re-formulating it as a minimization problem. Given a set of M centered snapshots $\{\mathbf{s}_k \in \mathbb{R}^n, k = 1, \ldots, M\}$, we have built two reduced order models: the revisited POD model, defined by the projector $\mathcal{P}(\mathbf{s})$ and its constrained counterpart, defined by the constrained projector $\mathcal{P}_c(\mathbf{s})$.

Let $\mathcal{E}$ be an Euclidean affine space defined on a vector space $E = \mathbb{R}^n$ with a frame $(O, \mathcal{B})$ where $\mathcal{B}$ is a canonical basis of E and endowed with the usual inner product. Let $\mathcal{F}$ be an affine subspace of $\mathcal{E}$, passing through $\mathbf{s}_0$ and of direction F of dimension $m \ll n$. Let $\mathbf{\Phi} = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_m]$ be an orthonormal basis of F. We have defined an orthogonal projection on $\mathcal{F}$ by $\mathcal{P} : \mathcal{E} \to \mathcal{F}$

$$\mathbf{s} \mapsto Argmin\left\{||\mathbf{s} - \mathbf{v}||^2, \quad \mathbf{v} \in \mathcal{F}\right\}. \tag{1}$$

Given a matrix $\mathbf{S}$ of $\boldsymbol{M}$ snapshots $\mathbf{s}_k \in \mathcal{E},\ k = 1, \ldots, M$ centered around $\mathbf{s}_0$

$$\mathbf{S} = [\mathbf{s}_1 - \mathbf{s}_0 \quad \ldots \quad \mathbf{s}_M - \mathbf{s}_0] \tag{2}$$

and considering, without loss of generality that $M < n$, we have shown that the best orthogonal projector $\mathcal{P}(\mathbf{s})$ minimizing the snapshot reconstruction error

$$J(\mathbf{\Phi}) = ||\mathcal{P}(\mathbf{S}) - \mathbf{S}||_F^2 / ||\mathbf{S}||_F^2, \tag{3}$$

where $||\cdot||_F^2$ is the Frobenius matrix norm and $\mathcal{P}(\mathbf{S})$ stands for projector $\mathcal{P}$ applied column-wise to the snapshot matrix $\mathbf{S}$

$$\mathcal{P}(\mathbf{S}) = [\mathcal{P}(\mathbf{s}_1) \quad \ldots \quad \mathcal{P}(\mathbf{s}_M)] \tag{4}$$

is given by the $m \leqslant M$ first eigenmodes of the system

$$\mathbf{C}_S\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Lambda}_\Phi, \quad \mathbf{\Phi}^T\mathbf{\Phi} = \mathbf{I}_m, \tag{5}$$