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Boundary conditions for first order macroscopic models of vehicular traffic in the presence of tollgates

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ABSTRACT

This paper presents a new approach to the modeling of boundary conditions for first order models of vehicular traffic in highways. The first step consists in deriving a model for the dynamics of the flow of vehicles. Simulations of the parameters lead to a detailed analysis of the qualitative properties of the model. Subsequently, for such model, the statement of initial-boundary value problems is deduced, with domain decomposition, for a tract of highway between tollgates.

1. Introduction

The modeling of vehicular traffic, as known [13,2], can be developed at different representation scales, namely the micro-scale, where the dynamic of all driver-vehicle subsystems is individually considered, and the macro-scale, suitable to provide the time and space evolution of the macroscopic flow quantities, typically local density and mean velocity. An intermediate link between these two scales is offered by the kinetic theory approach, where the dependent variable is a probability distribution over the micro-scale state of the interacting vehicles. The critical analysis proposed in [2] shows that none of the aforesaid scale is fully satisfactory in capturing the complexity of the system under consideration. Possibly multiscale approaches need to be developed.

More in details, models at the macro-scale are not fully consistent with the physics of the granular, intermittent, flow of vehicles. However, they can offer models with a simple structure useful for the applications. First order models [7,1] simply consist in the mass conservation equation properly closed by a phenomenological model referring the local mean velocity to the local density conditions, including, in some general cases, density gradients; nonlocal effects may also be considered [14]. The book by Kerner [16] illustrates how empirical data can be properly collected and interpreted with the attempt of breaking the complexity of the system under consideration. As an example, in some recent papers [18–21] car-following models considering the relationship between micro and macro variables and taking into consideration proper anticipation effects are considered.

The formal structure of first order models is as follows:

$$\partial_t \rho + \partial_x(\rho \mathcal{B}[\rho]) = 0, \quad \text{with} \quad v = \mathcal{B}[\rho], \tag{1.1}$$

where $\mathcal{B}$, which approximate the mean local velocity, is an operator to be properly determined according to models suitable to take into account the dynamics at the micro-scale, and the square brackets are used to denote that $\mathcal{B}$ is a functional of ρ; in simple cases, such as those treated in the following, it is a function of ρ and its space derivative.

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Moreover, each vehicle is modeled as a point, i.e. its length is negligible with respect to the length of the road, although a maximal density n_M corresponding to bumper-to-bumper packing situation is considered; $\rho = n/n_M$, is the dimensionless number density of vehicles, where the number density of vehicles n is referred to the maximal density n_M; $v = V/V_L$, is the mean velocity referred to the limit velocity V_L, namely the maximal velocity that a vehicle can technically reach in a certain road. This quantity is related to V_M (maximal mean velocity of vehicles in free flow conditions) by the relation $V_L = (1+\mu)V_M$ where μ is a positive constant valid in all environments ($0 < \mu < 1$). These quantities depend on time and space, namely $\rho = \rho(t,x)$ and $v = v(t,x)$, where t is a dimensionless time being referred to ℓ/V_M, being ℓ the length of the road and x the dimensionless space being referred to ℓ. Finally, local dimensionless flow is obtained by the relation $q(t,x) = \rho(t,x)\, v(t,x)$.

These models can be implemented for applications with special attention to networks [6,15,8], where the computational simplicity of first order models is a basic requirement to deal with the complexity of large systems. However, the modeling approach should take into account the following:

1. The closure $v = \mathcal{B}[\rho]$ should be related to the dynamics at the lower scale and to the quality of the road-environment conditions;
2. The application of the model is effective if suitable boundary conditions can be implemented corresponding to real traffic conditions such as tollgates, junctions, and traffic highlights.

The mathematical model proposed in [11] is specifically focused on the first issue to derive a model for the time–space dynamics of the density. The derivation includes a phenomenological model referring the local mean velocity to the local density and density gradient. However it is useful looking ahead to applications and to the validation of models based on empirical data. Accordingly, this paper derives from [11] a model for the flow and focuses on the implementation of boundary conditions corresponding to the presence of tollgates. The approach can be technically generalized to junctions and other devices of networks.

More precisely, Section 2 deals with the derivation from Eq. (1.1) of a model, where the flow, rather than the density, is the dependent variable. Section 3 presents the implementation of boundary conditions by a detailed analysis of the modification of the flow dynamics due to the presence of tollgates. Finally, Section 4 proposes a critical analysis towards further developments.

2. Derivation and qualitative properties of a model for the flow

As already mentioned in Section 1, it is convenient replacing the equation for the density by an equivalent model for the flow $q = q(t,x)$. In fact, this is a quantity of greater interest for the interpretation of the flow conditions due to the fact that empirical data for the flow are more accurate than those for the density. Moreover, the implementation of boundary conditions can be viewed, as we shall see, as an external action to control the flow of vehicles at tollgates rather than to control the density.

Let us briefly summarize the mathematical model proposed in [11] focusing on the case of negligible free flow conditions, namely when the quality of the road and of the environmental conditions are such that the velocity of vehicles reduces even at small densities. The said model can be written as follows:

$$\partial_t \rho + H^{\pm}(\rho,\, \partial_x\rho; \tilde{\alpha})\, \partial_x \rho = K^{\pm}(\rho,\, \partial_x\rho; \tilde{\alpha})\, \partial_{xx}\rho, \tag{2.1}$$

where $0 \leqslant \tilde{\alpha} < 1$ is a parameter which takes into account the quality of the road-environment. More precisely $\tilde{\alpha} = 0$ corresponds to the worse conditions and $\tilde{\alpha} = 1$ to the best "ideal" ones that might even not be reached in practical cases. Here and in the sequel, the superscript "$\pm$" is used as a compact form to indicate the two different cases of positive and negative density gradient; in fact the concept of perceived (or apparent) density (introduced in [9]) has been used in [11], by the introduction of local gradients of the density:

$$\rho_a^{+}(\rho, \partial_x\rho) = \rho + (1-\rho)\, tanh^2(\partial_x\rho), \quad \rho_a^{-}(\rho, \partial_x\rho) = \rho - \rho\, tanh^2(\partial_x\rho), \tag{2.2}$$

for positive and negative density gradient respectively, such that a closure condition of Eq. (1.1) is derived as the following expressions which depend on the local density, its gradient, and on aforesaid parameter $\tilde{\alpha}$:

$$v^{\pm}(\rho,\, \partial_x\rho; \tilde{\alpha}) = \frac{\tilde{\alpha}}{\tilde{\alpha} + (1-\tilde{\alpha}) \exp\left(\frac{(\rho_a^{\pm}(\rho,\partial_x\rho))^2}{1-\rho_a^{\pm}(\rho,\partial_x\rho)}\right)}. \tag{2.3}$$

The explicit expressions of the coefficients of Eq. (2.1) are as follows:

$$H^{\pm}(\rho,\, \partial_x\rho; \tilde{\alpha}) = v^{\pm}(\rho,\, \partial_x\rho; \tilde{\alpha})\left(\rho \frac{1 - v^{\pm}(\rho, \partial_x\rho; \tilde{\alpha})}{cosh^2(\partial_x\rho)} M^{\pm}(\rho,\, \partial_x\rho) + 1\right), \tag{2.4}$$