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Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions

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ABSTRACT

In this paper, a new method based on the Legendre wavelets expansion together with operational matrices of fractional integration and derivative of these basis functions is proposed to solve fractional partial differential equations with Dirichlet boundary conditions. The proposed method is very convenient for solving such boundary value problems, since the boundary conditions are taken into account automatically. Convergence of the two-dimensional Legendre wavelets expansion is investigated. Illustrative examples are included to demonstrate the validity and applicability of the presented wavelets method.

1. Introduction

Fractional differential equations are generalized from integer order ones, which are obtained by replacing integer order derivatives by fractional ones [1]. In comparison with integer order differential equations, the fractional differential equations show many advantages over the simulation of natural physical processes and dynamic systems [2–4]. The applications of fractional calculus have been demonstrated by many authors. For examples, fractional calculus is applied to model the nonlinear oscillation of earthquake [5], fluid-dynamic traffic [6], continuum and statistical mechanics [7], signal processing [8], control theory [9], and dynamics of interfaces between nanoparticles and subtracts [10].

Approximation by orthogonal family of functions have found wide application in science and engineering. The most frequency used orthogonal functions are sine–cosine functions, block pulse functions, Legendre, Chebyshev and Laguerre polynomials. The main advantage of using an orthogonal basis is that the problem under consideration is reduced to a system of linear or nonlinear algebraic equations [11]. This can be done by truncating series of orthogonal basis functions for the solution of the problem and using the operational matrices [11,12].

It is well known that we can approximate any smooth function by the eigenfunctions of certain singular Sturm–Liouville problems such as Legendre or Chebyshev orthogonal polynomials. In this manner, the truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation [13]. This phenomenon is usually referred to as "spectral accuracy" [13].

Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines. Wavelets are used in system analysis, optimal control, numerical analysis, signal analysis for waveform representation and segmentations, time–frequency analysis and fast algorithms for easy implementation [14].

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However, wavelets are just another basis set which offers considerable advantages over alternative basis sets and allows us to attack problems not accessible with conventional numerical methods. Their main advantages are as in [15]: the basis set can be improved in a systematic way, different resolutions can be used in different regions of space, the coupling between different resolution levels is easy, there are few topological constraints for increased resolution regions, the Laplace operator is diagonally dominant in an appropriate wavelet basis, the matrix elements of the Laplace operator are very easy to calculate and the numerical effort scales linearly with respect to the system size. Therefore, in the last two decades wavelets method have been applied for solving partial differential equations (PDEs) [16–25].

It is worth mentioning that Legendre wavelets have both of spectral accuracy and orthogonality and other mentioned properties about wavelets [15].

In this work, a numerical method based on the two-dimensional Legendre wavelets is proposed for solving fractional partial differential equations with Dirichlet boundary conditions. In the proposed method, both of the operational matrices of fractional integration and derivative are mutually employed to obtain numerical solutions of the mentioned problems. The proposed method is very convenient for solving such boundary value problems, since the boundary conditions are taken into account automatically. Convergence analysis of the two-dimensional Legendre wavelets is investigated. Numerical results demonstrate the efficiency of this Legendre wavelets method in solving partial differential equation.

The paper is organized as follows: In Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus. Multiresolution analysis and the Legendre wavelets are described in Section 3. The operational matrices of fractional integration and derivative for the Legendre wavelets are introduced in Section 4. In Section 5, convergence of the two-dimensional Legendre wavelets expansion is established. In Section 6, we apply the Legendre wavelets method for fractional partial differential equations. In Section 7, the proposed method is applied to several numerical examples. Finally a conclusion is given in Section 8.

2. Basic definitions

In this section we give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results.

Definition 2.1. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$ and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2.2. The Riemann–Liouville fractional integration operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu, \mu \geqslant -1$, is defined as:

$$(I^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{1}$$

Definition 2.3. The fractional derivative operator of order $\alpha > 0$ in the Caputo sense is defined as [26]:

$$(D_*^\alpha f)(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n. \end{cases} \tag{2}$$

where n is an integer, $t > 0$, and $f \in C_1^n$.

The useful relation between the Riemann-Liouvill operator and Caputo operator is given by the following expression:

$$(I^\alpha D_*^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0, \; n-1 < \alpha \leqslant n, \tag{3}$$

where n is an integer, $t > 0$, and $f \in C_1^n$.

For more details about fractional calculus see [26].

3. Multiresolution analysis and Legendre wavelets

In this section, we first briefly review the multiresolution analysis which will be used for constructing orthonormal wavelets and then introduce Legendre wavelets as a type of orthonormal wavelets.

3.1. Multiresolution analysis

A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ is defined as a set of closed subspaces V_j with $j \in \mathbb{Z}$ that exhibit the following properties: