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Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials

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ABSTRACT

Ever since 2012 when Srivastava et al. [22] introduced and initiated the study of many interesting fundamental properties and characteristics of a certain pair of potentially useful families of the so-called generalized incomplete hypergeometric functions, there have appeared many closely-related works dealing essentially with notable developments involving various classes of generalized hypergeometric functions and generalized hypergeometric polynomials, which are defined by means of the corresponding incomplete and other novel extensions of the familiar Pochhammer symbol. Here, in this sequel to some of these earlier works, we derive several general families of hypergeometric generating functions by applying (for example) some such combinatorial identities as Gould's identity, which stem essentially from the Lagrange expansion theorem. We also indicate various (known or new) special cases and consequences of the results presented in this paper.

1. Introduction and Definitions

As usual, throughout this paper we denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. In terms of the familiar (Euler's) Gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \int_0^\infty e^{-t}\, t^{z-1}\, dt & \big(\Re(z) > 0\big) \\ \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1}(z+j)} & \big(z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}\big), \end{cases} \tag{1}$$

$$\big(\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\};\ \mathbb{Z}^- := \{-1,-2,-3,\ldots\};\ \mathbb{N} := \{1,2,3,\ldots\}\big),$$

a generalized binomial coefficient $\binom{\lambda}{\mu}$ may be defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \quad (\lambda, \mu \in \mathbb{C}), \tag{2}$$

so that, in the special case when

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$$\mu = n \quad (n \in \mathbb{N}_0;\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}_0), \tag{3}$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the Pochhammer symbol given, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{4}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [27, p. 21 *et seq.*]).

The function $\Gamma(z)$ defined by (1), as well as its so-called incomplete versions, that is, the *incomplete Gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined, respectively, by

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt \quad \left(\Re(s) > 0;\ x \geqq 0\right) \tag{5}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt \quad \left(x \geqq 0;\ \Re(s) > 0 \text{ when } x = 0\right), \tag{6}$$

are known to play important rôles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, for example, [1,2,8,9,11,13–15,23–26,34,35]; see also [22] and the references cited therein).

In the year 2012, the following pair of potentially useful families of generalized incomplete hypergeometric functions was introduced and studied systematically by Srivastava et al. [22, p. 675, Eqs. (4.1) and (4.2)]:

$${}_p\gamma_q\left[\begin{matrix}(a_1,x), a_2, \ldots, a_p; \\ b_1, \ldots, b_q;\end{matrix} z\right] := \sum_{n=0}^\infty \frac{(a_1;x)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{7}$$

and

$${}_p\Gamma_q\left[\begin{matrix}(a_1,x), a_2, \ldots, a_p; \\ b_1, \ldots, b_q;\end{matrix} z\right] := \sum_{n=0}^\infty \frac{[a_1;x]_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{8}$$

where, in terms of the incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (5) and (6), the *incomplete* Pochhammer symbols

$$(\lambda;x)_\nu \quad \text{and} \quad [\lambda;x]_\nu \quad (\lambda, \nu \in \mathbb{C};\ x \geqq 0)$$

are defined as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C};\ x \geqq 0) \tag{9}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C};\ x \geqq 0), \tag{10}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C};\ x \geqq 0), \tag{11}$$

where $(\lambda)_\nu$ is the Pochhammer symbol given by (4).

In the widely-scattered literature on the subject of this paper, one can find several interesting generalizations of the familiar (Euler's) gamma function $\Gamma(z)$ defined by (1), as well as the corresponding generalizations and extensions of the Beta function $B(\alpha,\beta)$, the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_pF_q$. For example, for an appropriately bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ of essentially arbitrary (real or complex) numbers, Srivastava et al. [29, p. 243 *et seq.*] recently considered the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ given by

$$\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right) = \begin{cases} \sum\limits_{\ell=0}^\infty \kappa_\ell \frac{z^\ell}{\ell!} & (|z| < R;\ R > 0;\ \kappa_0 := 1) \\ \mathfrak{M}_0 z^\omega \exp(z)\left[1 + O\left(\frac{1}{|z|}\right)\right] & (|z| \to \infty;\ \mathfrak{M}_0 > 0;\ \omega \in \mathbb{C}) \end{cases} \tag{12}$$

for some suitable constants $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. Then, in terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ defined by (12), Srivastava et al. [29] introduced some remarkably deep generalizations of the extended Gam-