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A penalty-finite element approximation to a Signorini two-body contact problem in thermoelasticity

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ABSTRACT

In this work, a numerical approximation by the finite element method of a quasi-static, frictionless, unilateral contact problem between two thermoelastic bodies, in two dimensions, is considered. The approximation scheme is based on a penalty formulation used to replace the Signorini contact condition. Convergence results and error bounds are obtained and some numerical experiments presented.

1. Introduction

We consider in this paper the time evolution of the temperature and the displacement of two homogeneous and isotropic thermoelastic bodies that occupy, in their reference configurations, the two convex polygonal bounded domains $\Omega_l \subset \mathbf{R}^2$, $l = 1,2$, having boundaries $\partial\Omega_l$ divided into three mutually disjoint parts $\Gamma_{l0}, \Gamma_{lt}, \Gamma_{lc}$, such that, $\partial\Omega_l = \overline{\Gamma}_{l0} \cup \overline{\Gamma}_{lt} \cup \overline{\Gamma}_{lc}$, $\Gamma_{l0} \neq \emptyset$ and $\Gamma_{lc} \neq \emptyset$. They are in contact along the common part $\Gamma_{1c} = \Gamma_{2c} \equiv \Gamma_c$ but, as the system evolves, they may come apart on a portion of Γ_c and eventually get in contact again at the contact zone Γ_c (see Fig. 1). The bodies are held fixed on Γ_{l0} and tractions are zero on $\Gamma_{lt}, l = 1,2$. We assume that the evolution process of the system is quasi-static, the contact is frictionless and that interpenetration is not allowed. Under these conditions, the system of energy and elasticity equations is, for $l = 1,2$,

$$\theta_{lt} - \triangle\, \theta_l = -m_l \operatorname{div} \boldsymbol{u}_{lt} \quad \text{in} \quad \Omega_l,\ t > 0, \tag{1}$$

$$\operatorname{div} \boldsymbol{\sigma}_l(\boldsymbol{u}_l) = \mathbf{0} \quad \text{in } \Omega_l, \quad t > 0, \tag{2}$$

where

$$\boldsymbol{\sigma}_l(\boldsymbol{u}_l) = \lambda_l \operatorname{div} \boldsymbol{u}_l \mathbf{I} + 2\,\mu_l\, \epsilon_l(\boldsymbol{u}_l) - m_l\, \theta_l\, \mathbf{I},$$

$$\epsilon_l(\boldsymbol{u}_l) = 0.5(\nabla \boldsymbol{u}_l + (\nabla \boldsymbol{u}_l)^t),$$

with initial and boundary conditions

$$\boldsymbol{u}_l = \mathbf{0} \quad \text{on } \Gamma_{l0}, \quad \boldsymbol{\sigma}_l \boldsymbol{\nu}_l = \mathbf{0} \quad \text{on } \Gamma_{lt}, \quad t > 0, \tag{3}$$

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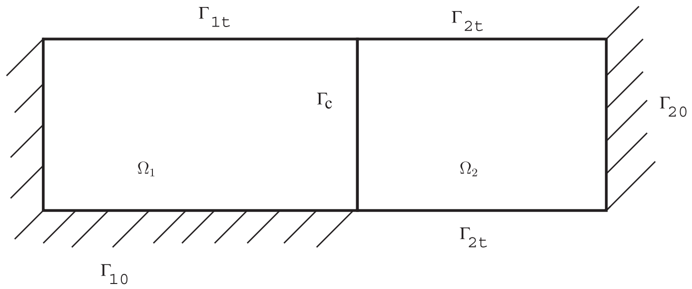

Fig. 1. The contact problem setting.

$$\left.\begin{array}{l} u_{1\nu}+u_{2\nu}\leqslant 0\\ \sigma_\nu\leqslant 0\\ \boldsymbol{\sigma}_{l\mathcal{T}}=\boldsymbol{\sigma}_l\boldsymbol{\nu}_l-\sigma_\nu\boldsymbol{\nu}_l=\mathbf{0}\\ \sigma_\nu(u_{1\nu}+u_{2\nu})=0\end{array}\right\}\quad \text{on } \Gamma_c,\quad t>0, \tag{4}$$

$$\theta_l=0\quad \text{on } \partial\Omega_l,\quad \theta_l(\mathbf{x},0)=\theta_{l0}(\mathbf{x}),\quad \mathbf{x}\in\Omega_l. \tag{5}$$

Here $\theta_l=\theta_l(\mathbf{x},t)$ represents temperature, $\boldsymbol{u}_l=\boldsymbol{u}_l(\mathbf{x},t)=(u_{l1}(\mathbf{x},t),\ u_{l2}(\mathbf{x},t))^t$ displacement with respect to the reference configuration and $\boldsymbol{\sigma}_l(\boldsymbol{u}_l)$ stress. For $l=1,2,\lambda_l,\ \mu_l>0$ are the Lamé constants, $m_l>0$ is the coefficient of thermal expansion, $\mathbf{I}$ is the 2×2 identity matrix, and $\operatorname{div}\boldsymbol{\sigma}_l$ is the 2-vector with ith component $\{\operatorname{div}\boldsymbol{\sigma}_l\}_i=\sum_{j=1}^2\boldsymbol{\sigma}_{lij,j}$. The index j that follows the comma indicates partial derivative with respect to $\mathbf{x}_j$. In (3) and (4) $\boldsymbol{\nu}_l=(\nu_{l1},\nu_{l2})^t$ is the normal unit vector pointing out of Ω_l, $u_{l\nu}=\boldsymbol{u}_l\cdot\boldsymbol{\nu}_l$ and $\sigma_\nu=(\boldsymbol{\sigma}_1\boldsymbol{\nu}_1)\cdot\boldsymbol{\nu}_1=(\boldsymbol{\sigma}_2\boldsymbol{\nu}_2)\cdot\boldsymbol{\nu}_2$ denotes the normal component of the stress. The first inequality in (4) is the Signorini contact condition of non-penetration. The other conditions state the action and reaction principle, the absence of friction, and that at the points where there is a gap between the bodies, the normal stress vanish.

Most of the work we found in the literature does not take into account the thermal effects. An static quasi-coupled (the term $-m_l\operatorname{div}\boldsymbol{u}_{lt}$ in (1) was removed) contact problem with friction and Signorini boundary condition in thermoelasticity was investigated by Hlaváček and Nedoma [8]. The existence and uniqueness of a solution was obtained and a finite element approximation analysed. Contact problems in thermoelasticity are often solved using the quasi-coupled theory and the coupling of heat and elasticity equations, considered in this article, introduces new challenges to the analysis. Quadratic finite elements were used by Hild and Laborde [7] to numerically approximate the solution of a two dimensional static Signorini contact problem between two elastic bodies. Convergence rates were obtained and numerical experiments carried out. A posteriori error estimates for two-body contact problems in elasticity were obtained by Wohlmuth [13]. The numerical approximation of frictionless, unilateral contact problems with boundary conditions of Signorin type, for plastic or elastic-viscoplastic bodies, was considered by Burguera and Viaño [2], Fernández, Hild and Viaño [5] and Han and Sofonea [6].

In those papers, the numerical approximations were based on variational inequalities modeling unilateral contact. Here, a penalty method is employed. This approach was used previously by Kikuchi and co-workers [9,10] to numerically approximate the solution of contact problems in linear elasticity. The main contribution of the present work is to propose and analyse a finite element approximation to the contact problem between two thermoelastic bodies, in two dimensions. We follow Copetti [4] where the unilateral contact, with initial gap, between an thermoelastic body and a deformable obstacle was studied. The analysis considered here can be extended to the three dimensional situation with adequate changes to the "triangulation" of the domains.

The paper is organized as follows: in Section 2, we introduce the finite element approximation and show that it has a unique solution. In Section 3, an error bound for the numerical approximation is provided and, finally, in Section 4, we present the results of some numerical experiments.

We consider the spaces $\mathbf{L}^2(\Omega_l)=\{L^2(\Omega_l)\}^2$, $\mathbf{H}^s(\Omega_l)=\{H^s(\Omega_l)\}^2$ and $\mathbf{H}^1_E(\Omega_l)=\{\boldsymbol{v}_l\in\mathbf{H}^1(\Omega_l)\,|\,\boldsymbol{v}_l=\mathbf{0}\text{ on }\Gamma_{l0}\}$ and denote the inner product in $\{L^2(\Omega_l)\}^n$, $n=1,2$, by $(\cdot,\cdot)$ and the norms of $\{L^2(\Omega_l)\}^n$ and $\{H^s(\Omega_l)\}^n$ by $\|\cdot\|$ and $\|\cdot\|_s$, respectively. As usual

$$\boldsymbol{\sigma}:\boldsymbol{\tau}=\sum_{i,j=1}^2\sigma_{ij}\tau_{ij}$$

for all 2×2 matrix-valued functions $\boldsymbol{\sigma},\boldsymbol{\tau}$. We endow $\mathbf{H}^1_E(\Omega_l)$ with the inner product

$$b(\boldsymbol{u}_l,\boldsymbol{v}_l)=\int_{\Omega_l}\boldsymbol{\epsilon}_l(\boldsymbol{u}_l):\boldsymbol{\epsilon}_l(\boldsymbol{v}_l)d\mathbf{x}$$

and denote the associated norm by $\|v_l\|_b=(b(\boldsymbol{v}_l,\boldsymbol{v}_l))^{\frac{1}{2}}$. As a consequence of Korn's inequality (see [1]),

$$\|\boldsymbol{v}_l\|_b\geqslant C\|\boldsymbol{v}_l\|_1\quad\forall\ \boldsymbol{v}_l\in\mathbf{H}^1_E(\Omega_l), \tag{6}$$