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Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems

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ABSTRACT

Using variational methods and critical point theory, we establish multiplicity results of nontrivial and nonnegative solutions for a perturbed fourth-order Kirchhoff type elliptic problem.

1. Introduction

Consider the following perturbed fourth-order Kirchhoff-type elliptic problem

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u + \rho|u|^{p-2}u = \lambda f(x,u) & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where $p > \max\{1, \frac{N}{2}\}$, $\lambda > 0$ is a real number, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ is a bounded smooth domain $\rho > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function.

The problem (1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

proposed by Kirchhoff [17] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Latter (2) was developed to form

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u). \tag{3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u). \tag{4}$$

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Problems like (4) can be used for modeling several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [1]. Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [10,12,14,22,24,27] and the references therein.

The fourth-order equation can describe the static form change of beam or the sport of rigid body. In [18], Lazer and Mckenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree theory, and variational methods and critical point theory, and we refer the reader to [3–6,8,9,13,15,16,19–21] and references therein.

In [29], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) & \text{in} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Also, in [30] the authors by using the mountain pass techniques and the truncation method studied the existence of nontrivial solutions for a class of fourth order elliptic equations of Kirchhoff-type.

In particular, in [23] employing a smooth version of Ricceri's variational principle [26], the authors ensured the existence of infinitely many solutions for the problem (1) when $\mu = 0$.

In the present paper, we obtain the existence of two solutions by combining an algebraic condition on f with the classical Ambrosetti-Rabinowitz condition.(AR): there exist $\nu > p$ and $R > 0$ such that

$$0 < \nu F(x,t) \leqslant tf(x,t), \quad \text{for all } |t| \geqslant R \quad \text{and} \quad \text{for all } x \in \Omega.$$

The role of (AR) is to ensure the boundness of the Palais–Smale sequences for the Euler–Lagrange functional associated to the problem. This is very crucial in the applications of critical point theory. The main tools used here are Theorems 2.1 and 2.2, which were improved in Theorem 2.1 of the paper [7] in which the authors, using variational methods, investigated the existence of infinitely many solutions for an autonomous elliptic Dirichlet problem involving the p-Laplacian.

For a through on the subject, we also refer the reader to [11].

2. Preliminaries

Our main tools are Theorems 2.1 and 2.2, consequences of a local minimum theorem [2, Theorem 3.1] which is inspired by Ricceri's variational principle (see [26]).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\vartheta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_1(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}, r_1 < r_2$, and

$$\rho_2(r) = \sup_{v \in \Phi^{-1}(]r, \infty[)} \frac{\Psi(v) - \sup\limits_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r \in \mathbb{R}$.

Theorem 2.1 [2, Theorem 5.1]. *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}, r_1 < r_2$, such that*

$$\vartheta(r_1, r_2) < \rho_1(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leqslant I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 [2, Theorem 5.3]. *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*