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Fractal generation using ternary 5-point interpolatory subdivision scheme

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ABSTRACT

This paper is devoted to explore the generation of fractal curves and surfaces using ternary 5-point interpolatory subdivision scheme. The fractal behavior of the limiting curves and surfaces is analyzed with the help of tension parameter. It may be noted that the proposed fractal scheme determines two intervals of range of shape control parameter. The behavior of the shape control parameter for the generation of fractals has been visualized through five examples which provides faster rate of generation of fractals.

1. Introduction

Subdivision is one of the most important gears of Computer Aided Geometric Design (CAGD). It is a simple way to create smooth curves and surfaces from discrete set of control points, that is the reason of its wide use in the field of geometrical measurement, computer graphics, reverse engineering, medical surgery simulations. Subdivision schemes can be characterized as interpolating and approximating subdivision schemes.

An important application of the subdivision schemes, is the generation of fractal curves. A geometric shape having symmetry of scale is termed as fractal,*i.e.* a shape zoomed in on a part of it an infinite number of times, still looks the same. Moreover, every part of a fractal is essentially a condensed-size copy of the whole shape. The astonishing thing about fractals is the assortment of their applications. They are used especially in computer modeling of irregular patterns and structures in nature. Benoit B. Mandelbrot (1924), 'father of fractals', inspected the relationship between fractals and nature. Properties of fractals comprise, self-similarity (repetition of patterns at all levels), infinite complexity and detail. That is the reason, fractals are widely used in real world problems like space–time, seismicity, sympathetic nerve discharge, natural human standing [15] etc.

de Rham, a French mathematician (1956) [1], is the poineer in the field of subdivision. He introduced the first corner cutting piecewise linear approximating subdivision scheme, generating C^1 limiting curve. In 1974, Chaikin [2] presented corner cutting subdivision scheme, generating C^1 limiting curve. Binary 3-point and ternary 3-point approximating subdivision schemes, generating C^3 and C^1 curves respectively, were proposed by Hassan and Dodgson [3] in 2002. Siddiqi and Nadeem [4] introduced a binary 3-point approximating subdivision scheme that generates C^2 curve, in 2007. Siddiqi and Rehan [5] developed a binary 4-point approximating subdivision scheme and new corner cutting scheme in 2010. The first interpolatory binary 4-point subdivision scheme, generating C^1 curve, was introduced by Dyn [6] et al. in 1987. Later on, different interpolatory subdivision schemes were designed. In 1989, Deslauries and Dubuc [7] designed a binary 4-point interpolating

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subdivision scheme generating C^1 limiting curve. Weissman [8] also offered a 6-point interpolating scheme in 1990. In 2002, Hassan and Dodgson [3] presented ternary three point interpolating subdivision scheme yielding C^1 continuous curve. In 2007, Beccari et al. [9] proposed an interpolating 4-point ternary nonstationary subdivision scheme with tension control that generates C^2 limiting curves. In 2009, Zheng [10] introduced 2n-1-point ternary interpolating subdivision schemes. Mustafa and Pakeeza [11] introduced a new 6-point ternary interpolating subdivision scheme generating C^2 limiting curve.

In 2007, Zheng et al. [12] proved that the limit curves generated by binary 4-point and ternary 3-point interpolatory subdivision schemes are fractals, keeping the corresponding tension parameters within certain specific ranges. Again in 2007 Zheng et al. [13] anticipated that the limit curves engendered by the ternary three point interpolating subdivision scheme with two parameters are fractal curves for some precise ranges of the parameters.

Jaun Wang et al. [16] discussed the fractal properties of the generalized Chaikin corner-cutting subdivision scheme on the basis of its properties of limit points, in 2011.

In this paper, fractal scheme introduced by Zheng et al. [13] is followed to view the fractal behavior conforming to the ternary 5-point interpolatory subdivision scheme proposed by Ghulam Mustafa, et al. [14]. Since the fractal scheme introduced in this paper uses more number of control points as compared to the scheme proposed by Zheng et al. [12], therefore this scheme offers a faster rate of generation of fractal curves and gives more degree of freedom to designer to have a control on the shape of fractal curve.

The ternary 5-point interpolating subdivision scheme is given as follows.

Given the set of initial control points $\mathbf{P}^0 = \left\{\mathbf{P}_i^0 \in \mathbf{R}^d\right\}_{i=-1}^{n+1}$. Let $\mathbf{P}^k = \left\{\mathbf{P}_i^k\right\}_{i=-1}^{3^k n+1}$ be the set of control points at level $k(k \geqslant 0, k \in \mathbf{Z})$, and $\left\{\mathbf{P}_i^{k+1}\right\}_{i=-1}^{3^k n+1}$ satisfy the following rules, recursively

$$\begin{cases} \mathbf{P}_{3i-1}^{k+1} = & (\omega - \frac{4}{81})\mathbf{P}_{i-2}^k + (-4\omega + \frac{10}{27})\mathbf{P}_{i-1}^k + (6\omega + \frac{20}{27})\mathbf{P}_i^k + (-4\omega - \frac{5}{81})\mathbf{P}_{i+1}^k + \omega\mathbf{P}_{i+2}^k, \quad 0 \leqslant i \leq 3^k, \\ \mathbf{P}_{3i}^{k+1} = & \mathbf{P}_i^k, \quad 0 \leqslant i \leq 3^k, \\ \mathbf{P}_{3i+1}^{k+1} = & \omega\mathbf{P}_{i-2}^k + (-4\omega - \frac{5}{81})\mathbf{P}_{i-1}^k + (6\omega + \frac{20}{27})\mathbf{P}_i^k + (-4\omega + \frac{10}{27})\mathbf{P}_{i+1}^k + (\omega - \frac{4}{81})\mathbf{P}_{i+2}^k, \quad 0 \leqslant i \leq 3^k. \end{cases} \tag{1}$$

where ω is the tension parameter.

Since for $-\frac{7}{81} < \omega < \frac{53}{324}$, the scheme is C^0 continuous. Similarly for $-\frac{1}{324} < \omega < \frac{2}{81}$ the scheme is C^1 continuous, and for $\frac{1}{324} < \omega < \frac{1}{162}$, the scheme is C^2 continuous. In the following section the relationship between tension parameter ω and the fractal behavior of the limit curve is analyzed.

2. Generation of fractal curves using ternary 5-point interpolatory subdivision scheme

Cogitate two arbitrary fixed control points $\mathbf{P}_i^k$ and $\mathbf{P}_j^k$ after n subdivision steps, where $\forall k \in \mathbf{Z},\ k \geqslant 0$. The effect of the parameter ω is needed to be analyzed on the sum of all the small edges between the two points after another m subdivision steps. For simplicity, the effect between the two initial control points, say, $\mathbf{P}_0^0$ and $\mathbf{P}_1^0$ is analyzed. According to the subdivision scheme (1), it is known that $\mathbf{P}_0^k \equiv \mathbf{P}_0^0$, where $k \geqslant 0$, and

$$\begin{cases} \mathbf{P}_1^{k+1} &= \omega\ \mathbf{P}_{-2}^k + (-\frac{5}{81} - 4\omega)\mathbf{P}_{-1}^k + (\frac{20}{27} + 6\omega)\mathbf{P}_0^k + (\frac{10}{27} - 4\omega)\mathbf{P}_1^k + (-\frac{4}{81} + \omega)\mathbf{P}_2^k, \\ \mathbf{P}_{-1}^{k+1} &= (-\frac{4}{81} + \omega)\mathbf{P}_{-2}^k + (\frac{10}{27} - 4\omega)\mathbf{P}_{-1}^k + (\frac{20}{27} + 6\omega)\mathbf{P}_0^k + (-\frac{5}{81} - 4\omega)\mathbf{P}_1^k + \omega\mathbf{P}_2^k. \end{cases} \tag{2}$$

Let

$$\mathbf{V}_k = \mathbf{P}_1^k - \mathbf{P}_0^k,$$

$$\mathbf{S}_k = \mathbf{P}_2^k - \mathbf{P}_1^k,$$

$$\mathbf{R}_k = \mathbf{P}_3^k - \mathbf{P}_2^k,$$

Then the difference equations, for the edge vectors $\mathbf{V}_k$, $\mathbf{S}_k$ and $\mathbf{R}_k$, can be obtained as follows:
Taking

$$\mathbf{U}_k = \mathbf{P}_1^k - \mathbf{P}_{-1}^k,\ \ \mathbf{W}_k = \mathbf{P}_0^k - \mathbf{P}_{-1}^k,\ \ \mathbf{Z}_k = \mathbf{P}_2^k - \mathbf{P}_{-2}^k. \text{ So } \mathbf{U}_k = \mathbf{V}_k + \mathbf{W}_k,$$

$\mathbf{U}_{k+1}$ can be written as

$$\mathbf{U}_{k+1} = \mathbf{P}_1^{k+1} - \mathbf{P}_{-1}^{k+1} = \frac{35}{81}\mathbf{U}_k - \frac{4}{81}\mathbf{Z}_k, \tag{3}$$

which can be rewritten as

$$\mathbf{U}_{k+1} - \frac{35}{81}\mathbf{U}_k = -\frac{4}{81}\mathbf{Z}_k. \tag{4}$$