Applied Mathematics and Computation 234 (2014) 486–505

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Explicit solutions to dynamic diffusion-type equations and their time integrals

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ARTICLE INFO

Keywords:
Diffusion equation
Heat equation
Semidiscrete equations
Partial dynamic equations
Time scales
Random walk

ABSTRACT

This paper deals with solutions of diffusion-type partial dynamic equations on discrete-space domains. We provide two methods for finding explicit solutions, examine their asymptotic behavior and time integrability. These properties depend significantly not only on the underlying time structure but also on the dimension and symmetry of the problem. Throughout the paper, the results are interpreted in the context of random walks and related stochastic processes.

1. Introduction

Semidiscrete partial differential equations have attracted attention of researchers in several applied areas where the discrete space occurs naturally, e.g. in biology [4], signal and image processing [13], and stochastic processes [7]. Nonetheless, to our knowledge there is no systematic theory of semidiscrete partial differential equations. In this work, we study solutions of partial dynamic equations of diffusion type on domains with discrete space and general time structure (continuous, discrete and other). We consider the equation

$$u^{\Delta_t}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{1.1}$$

where $a, b, c \in \mathbb{R}$ are constants and $\mathbb{T}$ is a time scale (arbitrary closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which coincides with the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, or the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x are not used, we omit the lower index t in u^{Δ_t} and write u^{Δ} only. The time scale calculus is used as a tool to obtain general results from which the corresponding statements for discrete and semidiscrete diffusion follow easily. Readers who are not familiar with the basic principles and notations of this theory are kindly asked to consult Stefan Hilger's original paper [10] or the survey [3]. This paper contributes to recent efforts of several researchers who have studied partial dynamic equations (e.g. [1,2,11,12,19]).

The present work is a free continuation of our recent paper [18], where we started to develop a systematic theory for equations of the form (1.1). Note that if $b = -2a = -2c$, the equation represents the space-discretized version of the classical diffusion equation (therefore, we talk about diffusion-type equations). Also, if $b = -a$ and $c = 0$ (or $b = -c$ and $a = 0$), we get the transport equation with discrete space. Another motivation for questions studied in this paper comes from the connection of (1.1) with Markov processes. Indeed, consider a one-dimensional discrete-time random walk on $\mathbb{Z}$. Let $p, q, r \in [0,1]$ be

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http://dx.doi.org/10.1016/j.amc.2014.01.176
0096-3003/

the probabilities of going left, standing still, and going right, respectively (so that $p+q+r=1$). If $u(x,t)$ is the probability of visiting point x at time t, we get $u(x,t+1)=pu(x+1,t)+qu(x,t)+ru(x-1,t)$, or the equivalent diffusion-type equation

$$u^{\Delta}(x,t)=pu(x+1,t)+(q-1)u(x,t)+ru(x-1,t),\quad x\in\mathbb{Z},\ t\in\mathbb{N}_0,$$

which, coupled with the initial condition $u(0,0)=1$ and $u(x,0)=0$ for $x\neq 0$, describes the random walk starting from the origin.

Next, consider a continuous-time random walk on $\mathbb{Z}$. Assume that in a time interval $[t,t+h]$, the probabilities of going left and right are $ph+o(h)$ and $rh+o(h)$, respectively. It follows that $u(x,t+h)=(ph+o(h))u(x,t+1)+(1-ph-rh+o(h))u(x,t)+(rh+o(h))u(x,t-1)$. By subtracting $u(x,t)$, dividing by h and passing to the limit $h\to 0$, we get the diffusion-type equation

$$u^{\Delta}(x,t)=pu(x+1,t)+(-p-r)u(x,t)+ru(x-1,t),\quad x\in\mathbb{Z},\ t\in\mathbb{R}_0^+.$$

Finally, for a general time scale $\mathbb{T}$, solutions of (1.1) can be regarded as heterogeneous stochastic processes. This interesting relationship is discussed throughout the paper and illustrates our results.

In Section 2, we briefly summarize the main results from [18]. In Section 3, we present two methods for finding explicit solutions of (1.1) once a particular time scale is given. These methods are then used in Section 4 to examine the asymptotic behavior of solutions as well as finiteness of their time integrals. We calculate the exact values of these integrals for $\mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=\mathbb{R}$, and discover the surprising fact that they coincide. In Section 5, multidimensional diffusion equations are briefly considered and we prove a slight generalization of Pólya's famous result [17] on the recurrence of symmetric random walks in $\mathbb{Z}^N$.

2. Preliminaries

Let us start with a short overview of the main results from [18], which will be used later. We consider dynamic diffusion-type equations of the form

$$u^{\Delta}(x,t)=au(x+1,t)+bu(x,t)+cu(x-1,t),\quad x\in\mathbb{Z},\ t\in\mathbb{T},\tag{2.1}$$

where $a,\ b,\ c$ are real numbers. The graininess of $\mathbb{T}$ influences the behavior of solutions in a substantial way, and some of the results presented in this section assume that the graininess is sufficiently small. The paper [18] contains a wealth of examples showing that these graininess conditions are indeed necessary.

The first result is an existence-uniqueness theorem for Eq. (2.1). Assume that X is a Banach space, $t_0\in\mathbb{T}$, and A is a bounded linear operator on X such that $I+A\mu(t)$ is invertible for every $t\in(-\infty,t_0)_{\mathbb{T}}$, where μ stands for the graininess function. Recall that the time scale exponential function $t\mapsto e_A(t,t_0)$ is defined as the unique solution of the initial-value problem

$$\begin{aligned} x^{\Delta}(t) &= Ax(t),\quad t\in\mathbb{T},\\ x(t_0) &= I. \end{aligned}\tag{2.2}$$

We use the symbol $\ell^{\infty}(\mathbb{Z})$ to denote the space of all bounded real sequences $\{u_n\}_{n\in\mathbb{Z}}$.

Theorem 2.1. *Consider an interval $[T_1,T_2]_{\mathbb{T}}\subset\mathbb{T}$ and a point $t_0\in[T_1,T_2]_{\mathbb{T}}$. Let $u^0\in\ell^{\infty}(\mathbb{Z})$. Assume that $\mu(t)<\frac{1}{|a|+|b|+|c|}$ for every $t\in[T_1,t_0)_{\mathbb{T}}$. Let the operator $A:\ell^{\infty}(\mathbb{Z})\to\ell^{\infty}(\mathbb{Z})$ be given by*

$$A(\{u_n\}_{n\in\mathbb{Z}})=\{au_{n+1}+bu_n+cu_{n-1}\}_{n\in\mathbb{Z}}.$$

Also, define the function $U:[T_1,T_2]_{\mathbb{T}}\to\ell^{\infty}(\mathbb{Z})$ by $U(t)=e_A(t,t_0)u^0,t\in[T_1,T_2]_{\mathbb{T}}$. Then

$$u(x,t)=U(t)_x,\quad x\in\mathbb{Z},\ t\in[T_1,T_2]_{\mathbb{T}},$$

is the unique bounded solution of Eq. (2.1) on $\mathbb{Z}\times[T_1,T_2]_{\mathbb{T}}$ such that $u(x,t_0)=u_x^0$ for every $x\in\mathbb{Z}$.

The superposition principle allows us to easily find explicit solutions for general initial conditions.

Theorem 2.2. *Let $u:\mathbb{Z}\times[t_0,T]_{\mathbb{T}}\to\mathbb{R}$ be the unique bounded solution of Eq. (2.1) corresponding to the initial condition*

$$u(x,t_0)=\begin{cases}1 & \text{if } x=0,\\ 0 & \text{if } x\neq 0.\end{cases}$$

If $\{c_k\}_{k\in\mathbb{Z}}$ is an arbitrary bounded real sequence, then

$$v(x,t)=\sum_{k\in\mathbb{Z}}c_k u(x-k,t)$$

is the unique bounded solution of Eq. (2.1) corresponding to the initial condition $v(x,t_0)=c_x,\ x\in\mathbb{Z}$.

The next theorem shows that for solutions of Eq. (2.1) with $a+b+c=0$, the sum $\sum_{x\in\mathbb{Z}}u(x,t)$ is the same for all t.