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Novel estimations for the eigenvalue bounds of complex interval matrices

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ABSTRACT

Our work proposes two methods that estimate the eigenvalue bounds (left/right for real and imaginary parts) of complex interval matrices. The first method expresses each bound as an algebraic sum of weighted matrix measures, where the measure corresponds to the spectral norm and the weighting matrix is diagonal and positive definite, with unknown entries. The optimization with respect to the entries of the weighting matrix yields the best value of the bound; the computational approach is ensured as the minimization of a linear objective function subject to bilinear-matrix-inequality constraints and interval constraints. The bounds are proved to be better than those provided by other estimation techniques. The second method constructs four real matrices so that each of them can be exploited as a comparison matrix for the complex interval matrix and allows the estimation of one of the eigenvalue bounds. The two methods we propose rely on different mathematical backgrounds, and between the resulting bounds no firm inequality can be stated; therefore, they are equally useful in applications, as reflected by the numerical case studies presented in the paper.

1. Introduction

Let the real matrices $A^-, A^+, B^-, B^+ \in \mathbb{R}^{n\times n}$ satisfy the componentwise inequalities $A^- \leqslant A^+, B^- \leqslant B^+$, and define the *real interval matrices*

$$\mathcal{A} = [A^-, A^+] = A_0 + [-R_A, R_A] = \left\{A \in \mathbb{R}^{n\times n} \middle| \quad A^- \leqslant A \leqslant A^+\right\} \subset \mathbb{R}^{n\times n}, \tag{1}$$

$$\mathcal{B} = [B^-, B^+] = B_0 + [-R_B, R_B] = \left\{B \in \mathbb{R}^{n\times n} \middle| \quad B^- \leqslant B \leqslant B^+\right\} \subset \mathbb{R}^{n\times n} \tag{2}$$

having $A_0 = \frac{1}{2}(A^- + A^+), B_0 = \frac{1}{2}(B^- + B^+)$ as their *centers*, and the nonnegative matrices $R_A = \frac{1}{2}(A^+ - A^-) \geqslant 0, R_B = \frac{1}{2}(B^+ - B^-) \geqslant 0$ as their *radii*.

The family (set) of complex matrices

$$\mathcal{C} = \mathcal{A} + j\mathcal{B} = \{C = A + jB| \quad A \in \mathcal{A}, \quad B \in \mathcal{B}\} \subset \mathbb{C}^{n\times n}, \tag{3}$$

where j is the imaginary unit, i.e. $j^2 = -1$, is called a *complex interval matrix*. For the description of $\mathcal{C}$ we are also going to use the complex matrices $C_0 = A_0 + jB_0 \in \mathbb{C}^{n\times n}$ and $R = R_A + jR_B \in \mathbb{C}^{n\times n}$, with $A_0, B_0, R_A, R_B \in \mathbb{R}^{n\times n}$ introduced above.

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For a matrix $C \in \mathcal{C}$, let $\lambda_k(C), k = 1, \ldots, n$, denote its *eigenvalues* (i.e. the roots of its *characteristic polynomial* $\chi_C(\lambda) = \det(\lambda I_n - C)$). Given an interval matrix $\mathcal{C} \subset \mathbb{C}^{n\times n}$, the union $\bigcup_{C\in\mathcal{C}} \left\{ \bigcup_{k=1,\ldots,n} \lambda_k(C) \right\} \subset \mathbb{C}^n$ defines the *eigenvalue-set* of $\mathcal{C}$.

The investigation of the location in $\mathbb{C}$ of the eigenvalue-set presents an intrinsic mathematical motivation and, concomitantly, is related to all research areas where interval matrices offer effective exploration tools. For instance, the evaluation of the convergence rates for numeric algorithms, or the analysis of the dynamic properties exhibited by uncertain linear systems are just two topics of already known interest for interval-matrix-based approaches.

The *exact bounds* of the eigenvalue-range of $\mathcal{C}$ are defined for the real parts by:

$$\mathfrak{R}^-(\mathcal{C}) = \min_{C\in\mathcal{C}} \min_{k=1,\ldots,n} Re\lambda_k(C), \quad \mathfrak{R}^+(\mathcal{C}) = \max_{C\in\mathcal{C}} \max_{k=1,\ldots,n} Re\lambda_k(C) \tag{4}$$

and for the imaginary parts by:

$$\mathfrak{I}^-(\mathcal{C}) = \min_{C\in\mathcal{C}} \min_{k=1,\ldots,n} Im\lambda_k(C), \quad \mathfrak{I}^+(\mathcal{C}) = \max_{C\in\mathcal{C}} \max_{k=1,\ldots,n} Im\lambda_k(C). \tag{5}$$

Generally speaking, the values of the exact bounds (4) and (5) are not available, and, therefore, different methods have been devised for estimating these unavailable values, as commented below. Our comments do not refer to the methods designed for symmetrical interval matrices, which are based on specialized results.

In the case of real interval matrices of form $\mathcal{A}$ (1), seminal ideas for estimating the exact bounds (4) and (5) are reported by the following works. [1] uses some symmetric and skew-symmetric constant matrices built from A_0 and R_A; [2] searches fixed points of a function constructed from A_0 and parameterized in accordance with R_A (the procedure refers only to the estimation of the right bound in (4)); [3] develops an evolutionary strategy; [4] applies the eigenvalue perturbation theory to A_0, taking into account the information given by R_A (the procedure refers only to the estimation of the bounds in (4)); [5] exploits the continuity of the characteristic polynomials (the procedure is suitable for sparse structures); [6] elaborates a filtering technique. A thorough comparative analysis of the methods in [1,2,4] accompanied by relevant numerical tests is presented by our recent article [7].

In the case of complex interval matrices of form $\mathcal{C}$ (3), estimations of the exact bounds (4) and (5) are proposed by the following papers. [8] utilizes some Hermitian and skew-Hermitian constant matrices built from C_0 and R (the procedure extends the scenario created by [1] for the real case); [9] adapts the estimation for interval matrices of general form to a framework that allows the use of algorithms existing for symmetric interval matrices, [10] relies on the concept of radii polynomials in order to apply the contraction mapping theory.

The current paper provides novel estimations of the exact bounds defined by (4) and (5). Two types of methods are considered, their principles being summarized below.

- The *first estimation principle* uses $A_0, B_0, R_A, R_B \in \mathbb{R}^{n\times n}$ to express each bound of $\mathcal{C}$ eigenvalue-range as an algebraic sum of weighted matrix measures, where the weighting matrix is diagonal and positive-definite. Each such a sum is optimized with respect to the weighting-matrix entries, and the numerical tractability is ensured as an extreme value problem, defined by a linear objective function subject to bilinear-matrix-inequality constraints and interval constraints.
- The *second estimation principle* uses $A^-, A^+, B^-, B^+ \in \mathbb{R}^{n\times n}$ to construct four real matrices which are exploited as comparison matrices for the whole family represented by $\mathcal{C}$. Each comparison matrix is proved to have a real eigenvalue that estimates one of the bounds of $\mathcal{C}$ eigenvalue-range.

The two methods exploit different mathematical backgrounds and, in general no inequality holds true between the resulting bounds – fact which means a "competition" between the two methods when studying an arbitrary interval matrix.

The current paper improves and enlarges the techniques used in our previous works [11–13]. Unlike Proposition 1 in [11], and Corollary 1 in [12], where the investigations are limited to the bounds of the real parts of the eigenvalues for real interval matrices, this paper covers both real and imaginary parts of complex interval matrices. Thus, the mentioned previous results are incorporated by the current paper as particular cases. On the other hand, this paper obtains better approximations for the values defined by (4) and (5), than reported in [13], as a consequence of refining the matrix-measure-based strategy for the majorization (respectively minorization) of the real and complex parts of the eigenvalues.

Besides the estimation improvement commented above relative to our previous work [13], we prove that the bounds provided by the first estimation principle in the current paper are better than those given in [1,8,9]. In order to refer to the estimated bounds proposed by the cited papers, as well as by the current paper, we are going to use the notations $\mathfrak{R}_\bullet^-(\mathcal{C}), \mathfrak{R}_\bullet^+(\mathcal{C}), \mathfrak{I}_\bullet^-(\mathcal{C}), \mathfrak{I}_\bullet^+(\mathcal{C})$, where the generic subscript "•" is particularized for the mentioned works, as follows "Hk" for [9], "Hz" for [8], "Ma" for [13], "Rn" for [1], "T2" and "T3" for Theorem 2, and, respectively Theorem 3 of the current paper. In all these cases, the estimation methods provide *outer bounds*, which, compared with the exact bounds defined by (4) and (5), satisfy the inequalities:

$$\mathfrak{R}_\bullet^-(\mathcal{C}) \leqslant \mathfrak{R}^-(\mathcal{C}) \leqslant \mathfrak{R}^+(\mathcal{C}) \leqslant \mathfrak{R}_\bullet^+(\mathcal{C}) \tag{6}$$

and, respectively,

$$\mathfrak{I}_\bullet^-(\mathcal{C}) \leqslant \mathfrak{I}^-(\mathcal{C}) \leqslant \mathfrak{I}^+(\mathcal{C}) \leqslant \mathfrak{I}_\bullet^+(\mathcal{C}). \tag{7}$$