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Can protection zone potentially strengthen protective effects in random environments?

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ABSTRACT

A biological population mechanism with a protection zone under random environments is described as a stochastic system in this paper. The long time dynamical properties, several methods to strengthen the effect of protection zone and a method of dividing the protection zone are given for this stochastic system. Results show that environmental influences on different parameters have different effects. Conclusions also illustrate the positive effect of protection zone. That is protection zone can potentially strengthen protective effects in random environments. At last, some examples and several numerical simulations are introduced to explain the theoretical conclusions.

1. Introduction

The method of establishing protection zone has been universally used and widely acknowledged as one of the most efficient approaches to protect the endangered species. Consequently, theoretical studies from the idea of mathematical modeling become more and more significant and popular. How is the effect if we take stochastic disturbance into a deterministic model? What factors will affect the effect of protection zone? We will study these topics by a stochastic mathematical model in this paper.

It is well known that the classical logistic growth equation is usually denoted by:

$$\mathrm{d}x(t) = x(t)(a - bx(t))\mathrm{d}t, \tag{1}$$

where $x(t)$ is the population density at time t. a and b are both positive constants. a represents the intrinsic growth rate of the population. a/b is the carry capacity of the environment. Eq. (1) models a single biological population whose members compete with each other for a limited amount of living space and food. However, biological populations are inevitably affected by environmental noises in the real world (see e.g. *Gard* [1,2]). Deterministic models exist some limitations in modeling ecosystems, for example, *Bandyopadhyay* and *Chattopadhyay* [3] pointed out that deterministic models are difficult to predict the future and to fitting data perfectly. Moreover, the parameters in the ecological systems (e.g. the birth rates, competition coefficients and carrying capacity coefficients) exhibit random fluctuations to some extent [4]. Considering the random fluctuations, scholars incorporate white noises in the deterministic models. Assume that fluctuations in the environments will affect all the parameters in Eq. (1), i.e. the intrinsic growth rate a and the intra-specific competition coefficient b. Scholars [5–13] usually use average values plus some error terms to estimate the white noises. That is

$$a \to a + \alpha_1 \quad \textit{noise}\ 1, \quad -b \to -b + \alpha_2 \quad \textit{noise}\ 2.$$

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Generally speaking, the error terms follow normal distributions by the well-known central limit theorem. So, the Itô-type stochastic model of Eq. (1) has the following form:

$$\mathrm{d}x(t) = x(t)(a - bx(t))\mathrm{d}t + \alpha_1 x(t)\mathrm{d}B_1(t) + \alpha_2 x^2(t)\mathrm{d}B_2(t), \tag{2}$$

where $B_i(t)$ $(i = 1, 2)$ is a 1-dimensional Brownian motion defined on a complete probability space $(\tilde{\Omega}, F, \{\mathcal{F}\}_{t\geqslant 0}, P)$. $\mathcal{F}_0$ contains all P-null sets, and $\{\mathcal{F}\}_{t\geqslant 0}$ is a right continuous filtration. α_i^2 represents the intensity of the white noises.

Now, we introduced a protected area for the stochastic model (2). The living region Ω is divided into two subregions Ω_1 and Ω_2. Ω_1 is the non-protected area and Ω_2 is a protected area in which there are plenty of food, medical security and catches or removal of biological resources are prohibited. Then the stochastic system with a protection zone becomes:

$$\begin{cases} \mathrm{d}x(t) = \left[x(t)(a - bx(t)) - \frac{D}{H}(x(t) - y(t)) - Ex(t)\right]\mathrm{d}t + \alpha_1 x(t)\mathrm{d}B_1(t) + \alpha_2 x^2(t)\mathrm{d}B_2(t) \\ \mathrm{d}y(t) = \left[y(t)(a - by(t)) + \frac{D}{h}(x(t) - y(t))\right]\mathrm{d}t + \alpha_1 y(t)\mathrm{d}B_1(t) + \alpha_2 y^2(t)\mathrm{d}B_2(t). \end{cases} \tag{3}$$

Here $x(t)$ and $y(t)$ are population densities in Ω_1 and Ω_2. D is the diffusion rate. $D(x - y)$ represents the diffusion capacity, i.e. the total biomass caused by diffusion effect. H is the size of Ω_1 and h is the size of Ω_2. E is the comprehensive negative effects of Ω_1 relative to Ω_2. Here, we assume all the parameters are positive constants. For a detailed model construction, we refer the reader to paper [14] which studied a deterministic biological population system with a protection zone.

The remaining parts of this paper are as follows: In Section 2, we show the existence and uniqueness of the positive solution. In Section 3, we give some long time dynamical properties, such as weakly persistent and extinction. In addition, some useful methods to strengthen the effect of protection zone and a method of dividing the protection zone are provided in this section. In Section 4, we use some examples and numerical simulations to illustrate the effects of protection zone.

2. Existence and uniqueness of the positive solution

As $x(t)$ and $y(t)$ in system (3) are population densities of Ω_1 and Ω_2 at time t, they should be nonnegative by their biological significance. Therefore, we are only interested in the positive solution. In this section, we will show the existence and uniqueness of the positive solution. In order to do this, we first show the following lemmas which are useful tools in the proof of the main results.

Lemma 1. *For any given initial value $x(0) = x_0 \in R_+ = \{x \in R : x > 0\}$, there is a unique solution $x(t)$ for Eq. (2) on $t \geqslant 0$ and the solution will remain in R_+ with probability one (almost surely or a.s.).*

Proof. This proof is motivated by the well-known works of *Mao* et al. [5] (Theorem 4.1). This method has been used in many documents (see e.g. [6–8,15]). The coefficients in Eq. (2) satisfy locally Lipschitz condition, then there is a unique maximal local solution $x(t)$ on $t \in [0, \tau_e]$ for any given initial value $x_0 \in R_+$, where τ_e is the explosion time (see [16,17]). Now, we need to prove $\tau_e = \infty$. Choose $k_0 > 0$ be sufficiently large such that x_0 lies within the interval $[1/k_0, k_0]$. For each positive integer $k > k_0$, define a stopping time as follows:

$$\tau_k = \inf\{t \in [0, \tau_e] : x(t) \notin (1/k, k)\}.$$

For the empty set $\emptyset$, we set $\inf \emptyset = \infty$. Clearly, τ_k is increasing as $k \to \infty$. Denote $\tau_\infty = \lim_{k\to+\infty}\tau_k$, then $\tau_\infty \leqslant \tau_e$ *a.s.* Now, we are at the position to show $\tau_\infty = \infty$. If this statement is not true, there exists a pair of constants T and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty < \infty\} > \varepsilon$. Thus, there exists an integer $k_1 \geqslant k_0$ satisfies

$$P\{\tau_k < T\} > \varepsilon, \ k > k_1. \tag{4}$$

Define a nonnegative function $V : R_+ \to R_+$ as:

$$V(h) = \sqrt{h} - 1 - \frac{1}{2}\ln h.$$

From the Itô formula, we get

$$\begin{aligned} \mathrm{d}V(x) = V_x\mathrm{d}x + 0.5V_{xx}(\mathrm{d}x)^2 &= (0.5x^{-0.5} - 0.5x^{-1})\left[x(a - bx)\mathrm{d}t + \alpha_1 x\mathrm{d}B_1(t) + \alpha_2 x^2\mathrm{d}B_2(t)\right] \\ &\quad + 0.5(-0.25x^{-1.5} + 0.5x^{-2})(\alpha_1^2 x^2 + \alpha_2^2 x^4)\mathrm{d}t \\ &= \left[-0.125\alpha_2^2 x^{2.5} + 0.25\alpha_2^2 x^2 - 0.5bx^{1.5} + 0.5bx + (0.5a - 0.125\alpha_1^2)x^{0.5} - 0.5a + 0.25\alpha_1^2\right]\mathrm{d}t \\ &\quad + 0.5\alpha_1(x^{0.5} - 1)\mathrm{d}B_1(t) + 0.5\alpha_2(x^{1.5} - x)\mathrm{d}B_2(t). \end{aligned}$$

Here, we write $x(t)$ as x for convenience's sake. Since a, b, α_1^2 and α_2^2 are all positive constants, there exists a constant $M_1 > 0$ such that

$$-0.125\alpha_2^2 x^{2.5} + 0.25\alpha_2^2 x^2 - 0.5bx^{1.5} + 0.5bx + (0.5a - 0.125\alpha_1^2)x^{0.5} - 0.5a + 0.25\alpha_1^2 < M_1.$$

Substitute the above inequality into Eq. (4), we conclude that