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Improved stability conditions for a class of stochastic Volterra–Levin equations

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ABSTRACT

In this paper, we study the mean square asymptotic stability of a class of generalized nonlinear stochastic Volterra–Levin equations by using fixed point theory. Several sufficient conditions are established for ensuring that the equation is mean square asymptotically stable as well as exponentially stable. The main results are new which generalize and improve some well-known results in Burton (2006) [4] and Luo (2010) [18]. Finally, two examples are given to illustrate our results.

1. Introduction

Volterra's equation turned out to be very important, which has been used to model the circulating fuel nuclear reactor, the neutron density and the neural networks, to name just a few applications.

In 1928, Volterra [1] modeled a biological problem with the following equation.

$$x'(t) = -\int_{t-L}^{t} a(t-s)g(x(s))ds, \quad \psi : [-L, 0] \to R, \tag{1}$$

where $L > 0$, $xg(x) > 0$ when $x \neq 0$, $a : [0, L] \to R$. Levin [2] obtained the asymptotical stability of the Eq. (1) under assumption:

$$a(t) \in C[0,\infty), \quad a(t) \geqslant 0, \quad a'(t) \leqslant 0, \quad a''(t) \geqslant 0, \quad \text{and} \quad a'''(t) \leqslant 0. \tag{2}$$

Together with Nohel, Levin [3] almost found the way to construct a Liapunov functional on the solution of Volterra's problem under assumption:

$$a(L) = 0, \quad a'(t) \leqslant 0, \quad a''(t) \geqslant 0, \quad \text{and} \quad a''(t)\, not \equiv 0. \tag{3}$$

It is clear that assumptions (2) and (3) are difficult to verified in real-world problems, but they are essential in the above argument. By asking more of function g, Burton [4] proved the solution tends to zero by using the fixed point method to the following equation.

$$x'(t) = -\int_{t-L}^{t} p(s-t)g(x(s))ds, \quad \psi : [-L, 0] \to R, \tag{4}$$

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where $L > 0$, $xg(x) > 0$ when $x \neq 0$, $p \in C([-L, 0], R)$ satisfying

$$\int_{-L}^{0} p(s)ds = 1. \tag{5}$$

For all $x, y \in R$, function $g : R \to R$ satisfies

$$|g(x) - g(y)| \leqslant K|x - y|, \quad \lim_{x\to 0} \frac{g(x)}{x} \text{ exists} \tag{6}$$

and

$$\frac{g(x)}{x} \geqslant \gamma \text{ for some constant } \gamma > 0. \tag{7}$$

Theorem 1.1 (Burton [4]). *Suppose (5) and (6) hold. If*

$$\lambda := 2K \int_{-L}^{0} |sp(s)|ds < 1, \tag{8}$$

then every solution of (4) is bounded. If (7) also holds, then every solution tends to zero.

The fixed point technique has been used widely, we referee the reader to [5–17] and therein. Here we highlight Luo's work [18]. Consider

$$\begin{cases} dx(t) = -\left(\int_{t-L}^{t} p(s-t)g(x(s))ds\right)dt + \Sigma(t)dB(t), & t > 0, \\ x(t) = \psi(t), & t \leqslant 0. \end{cases} \tag{9}$$

By the fixed points method, Luo gave conditions for exponential stability in mean square and almost sure exponential stability.

Theorem 1.2 (Luo [18]). *Suppose that the conditions (5)–(8) hold. Moreover, if one of the following two conditions holds:*

(i) $\int_0^t e^{2\gamma s}\Sigma^2(s)ds\, a.s.$ *is bounded for all* $t \geqslant 0$,

(ii) $\int_0^\infty e^{2\gamma s}\Sigma^2(s)ds = \infty$ *and* $e^{\frac{\gamma}{2}t}\Sigma^2(t) \to 0$ *as* $t \to \infty$, *then Eq. (9) is exponentially stable in mean square, that is,* $e^{\frac{\gamma}{2}t}E|x(t)|^2 \to 0$ *as* $t \to \infty$.

In [19], Li and Xu generalized the equation to a impulsive stochastic Volterra–Levin equation, and proved the existence and global attractivity of periodic solution for the equations by establishing a new integral inequality. Guo and Zhu [20] consider the effect of Poisson jumps on stability of stochastic Volterra–Levin equations by using the fixed point approach. More results about generalized stochastic differential equation, the reader can see [21–23] and references cited therein.

In fact, the assumption $\lambda := 2K\int_{-L}^{0}|sp(s)|ds < 1$ is vital to all the results obtained in [4,18–20]. An interesting question deserving further investigation is what will happen if $2K\int_{-L}^{0}|sp(s)|ds \geqslant 1$. Motivated by this question, by employing the average method and fixed point theorem, we focus on establishing the criteria to guarantee the asymptotical stability and exponential stability in mean square of the stochastic Volterra–Levin equation, which generalizes and improves the known results.

In this paper, we investigate the stability of stochastic Volterra–Levin equations of the form

$$\begin{cases} dx(t) = -\left(\int_{t-L}^{t} p(s-t)g(s, x(s))ds\right)dt + \Sigma(t)dB(t), & t > t_0, \\ x(t) = \psi(t), & t_0 - L \leqslant t \leqslant t_0, \end{cases} \tag{10}$$

where $p \in C([-L, 0], R)$, $g \in C(R \times R, R)$ and $\Sigma \in C(R, R)$ under assumptions:

(H1) $\int_{-L}^{0} p(s)ds = 1$.

(H2) $g(t, 0) = 0, xg(t, x) \geqslant 0$ and $\lim_{x\to 0}\frac{g(t,x)}{x} = \chi(t) < \infty$.

(H3) There are positive functions $\beta \in C(R, R)$ and $Q \in C(R, R)$ such that

$$\frac{g(t, x)}{x} \geqslant \beta(t)$$

and

$$|g(t, x) - g(t, y)| \leqslant Q(t)|x - y| \quad \text{for all } x, y \in R.$$

(H4) There exists a positive constant K such that $Q(t) \leqslant K$.