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Hamburger moment problem and Maximum Entropy: On the existence conditions

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ABSTRACT

The existence of Maximum Entropy solution for the reduced Hamburger moment problem is reconsidered. Existence conditions, previously appeared in literature, are revisited allowing an easy way to identifying the existence of Maximum Entropy solution. The obtained results suggest that, except for special sequences of moments unknown a priori, the Maximum Entropy solution for the non symmetric reduced Hamburger moment problem exists. For practical purposes, the replacing of the support $\mathbb{R}$ with a large enough finite interval finds a theoretical warranty. The symmetric case may be formulated as follows: once assigned the first $2M$ moments, if MaxEnt density does not exist (conclusion drawn uniquely from numerical evidence), MaxEnt density with the first $2M-2$ moments exists. In such a case, even if the first $2M$ moments are known, we have to settle for a density which carries less information. Theoretical results are illustrated through some numerical examples.

1. Introduction

The reduced Hamburger moment problem consists of recovering an unknown probability density function (pdf) $f(x)$, with support $\mathbb{R}$, from the knowledge of its associated sequence $\{\mu_j\}_{j=1}^{2M}$ of moments [1, p.28], with

$$\mu_j = \int_{\mathbb{R}} x^j f(x)dx, \quad j = 1,\ldots,2M, \;\; \mu_0 = 1. \tag{1.1}$$

The recovering of f from $\{\mu_j\}_{j=1}^{2M}$ is not unique and we call D^{2M} the set of solutions such that

$$\mu(f) =: \int_{\mathbb{R}} x^j f(x)dx, \quad j = 0,\ldots,2M, \;\; \forall f \in D^{2M}. \tag{1.2}$$

A common way to regularize the problem is the Maximum Entropy (MaxEnt) technique in which a solution of (1.2) is obtained maximizing the Shannon-entropy $H[f] = -\int_{\mathbb{R}} f(x)\ln f(x)dx$ under the constraint (1.1). It is well-known that a pdf of the form

$$f_{2M}(x) = \exp\left(-\sum_{j=0}^{2M} \lambda_j x^j\right) =: \exp_\lambda \tag{1.3}$$

has the largest entropy of all functions having the first $2M$ moments in common with f. Here $\boldsymbol{\lambda} = (\lambda_0,\ldots,\lambda_{2M})$ is the vector of Lagrange multipliers, with $\lambda_{2M} \geqslant 0$ to guarantee integrability. f_{2M} is supplemented by

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$$\mu_j = \int_{\mathbb{R}} x^j f_{2M}(x)dx, \quad j = 0, \ldots, 2M. \tag{1.4}$$

When $\lambda_{2M} > 0$, higher moments μ_{2M+1} and μ_{2M+2} are obtained integrating (1.4) by parts

$$2\mu_1 = \sum_{j=1}^{2M} j\lambda_j\mu_{j+1}, \quad 3\mu_2 = \sum_{j=1}^{2M} j\lambda_j\mu_{j+2}. \tag{1.5}$$

The required integrability of $\exp_\lambda$ restricts the multipliers vector to the set

$$\Lambda = \{\lambda \in \mathbb{R}^{2M+1} : \exp_\lambda \in L^1(\mathbb{R})\}$$

For $\boldsymbol{\lambda} \in \Lambda$ the moments of $\exp_\lambda$ in any order are well defined so that the collection of integrable exponential densities

$$E^{2M} = \{\exp_\lambda : \lambda \in \Lambda\}$$

is a subset of D^{2M}. In general $\mu(D^{2M})$ (the interior of moment space) will include strictly $\mu(E^{2M})$ (the moment space relative to the MaxEnt densities). Therefore, there are admissible moment vectors $\mu \in \mu(D^{2M})$ for which the moment problem (1.1) is solvable, but the MaxEnt problem (1.4) has no solution. The procedure in [2] to obtain $\mu(E^{2M}) \subset \mu(D^{2M})$ is reported below step by step.

- Pick any $\lambda \in \Lambda \cap \partial\Lambda$ ($\partial\Lambda$ indicates the boundary of Λ). $\lambda \in \Lambda \cap \partial\Lambda$ implies highest component $\lambda_{2M} = 0$.
- Calculate the moment vector $\mu_\lambda = \mu(\exp_\lambda)$.
- Add any positive number to the highest component $\mu = \mu_\lambda + \epsilon e_{2M+1}$, $\epsilon > 0$.

Then $\mu \in \mu(D^{2M})$ is an admissible vector, but the MaxEnt problem with constraints μ has no solution. After introducing the order relation

$$(u_0, \ldots, u_{2M}) \geqslant (v_0, \ldots, v_{2M}) \iff u_0 = v_0, \ldots, u_{2M-1} = v_{2M-1}, u_{2M} \geqslant v_{2M}$$

[2, Theorem 4] sounds as follows.
Theorem 1 [2, Theorem 4].

$$\mu(D^{2M}) \setminus \mu(E^{2M}) = \{\mu : \mu > \mu(\exp_\lambda), \ \lambda \in \Lambda \cap \partial\Lambda\}.$$

In particular, MaxEnt problem is solvable if and only if $\mu \in \mu(D^{2M})$ *satisfies* $\mu \ngtr \mu(\exp_\lambda)$ *for all* $\lambda \in \Lambda \cap \partial\Lambda$.

Integrability of $\exp_\lambda$ follows certainly when the highest Lagrange multiplier is positive. However, also in the case $\lambda_{2M} = 0$, integrability is possible provided $\lambda_{2M-1} = 0$ and $(\lambda_0, \ldots, \lambda_{2M-2})$ ensures integrability. Then the author [2] finds recursively for an arbitrary $2M$

$$\Lambda^{2M} = \{(\lambda_0, \ldots, \lambda_{2M}) : \lambda_{2M} > 0\} \cup \{(\lambda_0, \ldots, \lambda_{2M-2}, 0, 0) : (\lambda_0, \ldots, \lambda_{2M-2}) \in \Lambda^{2M-2}\}.$$

In particular, the crucial set $\Lambda^{2M} \cap \partial\Lambda^{2M}$ is given by

$$\Lambda^{2M} \cap \partial\Lambda^{2M} = \{(\lambda_0, \ldots, \lambda_{2M-2}, 0, 0) : (\lambda_0, \ldots, \lambda_{2M-2}) \in \Lambda^{2M-2}\}.$$

Now the actual problem in MaxEnt setup is: once given $\{\mu_j\}_{j=1}^{2M}$ does the MaxEnt density f_{2M} exist? Evidently the answer $\mu \ngtr \mu(\exp_\lambda)$ for all $\lambda \in \Lambda \cap \partial\Lambda$ provided by Theorem 1 and by the above recursive procedure on Λ^{2M} does not much help the practitioners of MaxEnt technique. Our purpose in this paper amounts to providing an answer to the above problem. In our main result we reformulate Theorem 1 through a constructive procedure enabling to give a quick answer to the existence of MaxEnt density. The obtained results are summarized in the here below Theorem 2 as follows

1. In the non symmetric reduced Hamburger moment problem, once $\{\mu_j\}_{j=1}^{2M} \in \mu(D^{2M})$ are assigned, the existence of a MaxEnt solution f_{2M} is guaranteed except for special sequences of moments unknown a priori. So that, excluding the latter ones, the positivity of the Hankel determinants, which is necessary condition of the solvability of the reduced Hamburger moment problem, guarantees the existence of f_{2M}. The results here obtained provide a theoretical background to the usual practice, in the approximation framework, which amounts to replacing the support $\mathbb{R}$ with an arbitrarily large interval $[a, b]$. Then the problem is numerically solved within a proper interval $[a, b]$, changing the original Hamburger problem into Hausdorff moment problem. In MaxEnt setup the latter admits a solution [2, Theorem 2] for each sequence $\{\mu_j\}_{j=1}^{2M} \in \mu(D^{2M})$. Then we might be led to the unpleasant fact that a MaxEnt solution always exists, although the original Hamburger moment problem does not admit any solution. From the main result here obtained such an event seems rare, so that the above practice has low probability to fail and may be considered a suitable tool to recover a probability density.
2. The symmetric reduced Hamburger moment problem (i.e. $\mu_{2j-1} = 0, 1 \leqslant j \leqslant M$) sounds as follows: if from numerical evidence the MaxEnt density f_{2M} does not exist, f_{2M-2} exists. Then we settle for f_{2M-2}, although the first $2M$ moments are known.