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The Optimal Homotopy Asymptotic Method for solving Blasius equation

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ABSTRACT

Starting from the reality that many known methods fail in the attempt to obtain analytic solutions of Blasius-type equations, in this work, a new procedure namely Optimal Homotopy Asymptotic Method (OHAM) is proposed to obtain an explicit analytical solution of the Blasius problem. Comparison with Howarth's numerical solution, as well as the obtained residual, reveals that the proposed method is highly accurate. This proves the validity and great potential of the proposed procedure (OHAM) as a new kind of powerful analytical tool for nonlinear problems.

1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear differential equations. Many powerful methods have been presented. Perturbation techniques have been widely applied to solve nonlinear problems, but like other analytical techniques, perturbation methods have their limitations: they are based on such assumption that a small parameter must exist [1]. This so-called small parameter assumption greatly restricts applications of perturbation techniques, because many nonlinear problems have no small parameters at all. Moreover, even if such a small parameter exists, the corresponding perturbation approximations are valid generally only for small values of this parameter and become useless as the value of the parameter increases.

For instance, consider the two-dimensional laminar viscous flow past a semi-infinite flat plate, governed by a non-linear ordinary differential equation [2]:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \quad \eta \in [0, \infty], \tag{1}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{2}$$

where the prime denotes the derivative with respect to the similarity variable which is defined as $\eta = y\sqrt{\frac{U}{vx}}$, the dimensionless function $f(\eta)$ is related to the stream function $\psi(x, y)$ by $f(\eta) = \frac{\psi}{\sqrt{vxU_\infty}}$. Here U_∞ is the constant velocity of the mainstream at infinity, v is the kinematic viscosity coefficient and x and y are two independent variables.

Note that the Blasius equation (1) is a special case of the so-called Falkner–Skan-equation

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$$f'''(\eta) + \alpha f(\eta) f''(\eta) + \beta[1 - f'(\eta)] = 0,$$

proposed by Falkner and Skan [3], which was studied by Howarth [4] and Schlichting [5]. It is well known that the Blasius equation is the mother of all boundary-layer equations in fluid mechanics.

In 1908, Blasius [2] provides a power series solution:

$$f(\eta) = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{3}$$

where

$$A_0 = A_1 = 1, \quad A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}, \quad (k \geqslant 2), \tag{4}$$

with $\binom{m}{n} = \frac{m!}{n!(m-n)!}$. Note that the expression (3) is not closed, because $\sigma = f''(0)$ is unknown. By means of matching two different approximations at a proper point, Blasius obtained the numerical result $\sigma = 0.332$. In 1938, Howarth [4] gained a more accurate value $\sigma = 0.33206$ by means of a numerical technique. However, by means of $\sigma = 0.33206$, $f(\eta)$ given by Eq. (3) is valid in a small region $0 \leqslant \eta \leqslant \rho_0 \approx 5.690$. Blasius' power series (3) is fundamentally an analytic-numerical solution, because the value of σ is gained by numerical techniques.

Until now lots of analytical methods were proposed to solve Blasius equation. He [6–7] proposed a perturbation technique coupled with an iteration technique. Comparison with Howarth's numerical solution reveals that the proposed method leads to the approximate value $\sigma = 0.3296$ with 0.73% accuracy. Asaithambi [8] found this number correct to nine decimal positions as $\sigma = 0.332057336$. The variational iteration method is applied for a reliable treatment of two forms of Blasius equation which comes from boundary layer equation by Wazwaz [9]. The same author proposed earlier a modified form of the Adomian decomposition method, which is found to be fast and accurate [10]. Wang [11] employed an algorithm based mainly on applying the Adomian decomposition method (ADM) to the transformation of the Blasius equation and later, Hashim [12] improved the numerical solution of Wang using the ADM–Pade approach. Recently, Fazio [13] solved numerically the Blasius problem, and its variants and extensions, by initial value methods derived within scaling invariance theory. Sinc-collocation method, which is a procedure converging to the solution at an exponential rate, is applied in [14] and the Homotopy Analysis Method (HAM) is successfully applied by Yao and Chen in [15] and Liao in [16]. Also, Yun [17] proposed an intuitive approach to the approximate analytical solution for the Blasius problem in the form of a logarithm of the hyperbolic cosine function.

All these prove that although the Blasius problem is a century old, it is still a topic of active current research.

The purpose of the present investigation is to obtain an explicit analytic solution of the Blasius equation by using a recently proposed technique called the Optimal Homotopy Asymptotic Method (OHAM) [18], which proved to be a powerful tool for solving strongly nonlinear problems. This technique was already used to solve various nonlinear problems arising in heat transfer, fluid dynamics and vibration [18–23] and it was shown that its main feature is a rigorous and reliable procedure to control the convergence of approximate solutions.

2. Basic idea of OHAM

We apply the Optimal Homotopy Asymptotic Method to the following differential equation:

$$N(f(\eta)) = 0, \quad B\left(f, \frac{df}{d\eta}\right) = 0, \tag{5}$$

where η is an independent variable, $f(\eta)$ is an unknown function, $N(f(\eta))$ is a nonlinear operator and B is a boundary operator.

By means of the OHAM one first construct a family of equations:

$$(1-p)[L(\varphi(\eta,p))] = H(p,\eta)[N(\varphi(\eta,p))], \quad B\left(\varphi(x,p), \frac{\partial \varphi(x,p)}{\partial x}\right) = 0, \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, L is a linear operator which depends on the boundary operator B and on the initial approximation f_0, $H(p,\eta)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0,\eta)$, $\varphi(\eta,p)$ is an unknown function, respectively. Obviously, when $p = 0$ and $p = 1$ it holds

$$\varphi(\eta,0) = f_0(\eta), \quad \varphi(\eta,1) = f(\eta), \tag{7}$$

respectively. Thus, as p increases from 0 to 1, the solution varies from the initial approximation $f_0(\eta)$ to the solution $f(\eta)$, where $f_0(\eta)$ is obtained from Eq. (6) for $p = 0$:

$$L(f_0(\eta)) = 0, \quad B\left(f_0, \frac{df_0}{d\eta}\right) = 0, \tag{8}$$

We have a great freedom to choose the so-called auxiliary operator L. More important, the traditional way to construct a homotopy cannot provide the most convenient way to adjust the convergence region and rate of approximate series.