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Half-discrete Hilbert-type inequalities with mean operators, the best constants, and applications

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ABSTRACT

In this article we derive several new half-discrete Hilbert-type inequalities with a general homogeneous kernel, involving arithmetic, geometric and harmonic mean operators. The main results are proved for the case of non-conjugate exponents. A special emphasis is given to determining conditions under which these inequalities include the best possible constants. As an application, we consider some operator expressions closely connected to established inequalities.

1. Introduction

Although classical, the Hilbert inequality (see [5]) is still of interest to numerous mathematicians. Nowadays, more than a century after its discovery, considerable attention is given to establishing such inequalities, where the functions and sequences are replaced with certain integral and discrete operators.

Throughout this article, we deal with Hilbert-type inequalities with a homogeneous kernel. Recall that a function $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is said to be homogeneous of degree $-s, s>0$, if $K(tx,ty)=t^{-s}K(x,y)$, for every $x,y,t\in\mathbb{R}_+$. In addition, for such a function we define

$$k(\eta)=\int_0^\infty K(1,t)t^{-\eta}dt.$$

If nothing else is explicitly stated, we assume that the integral $k(\eta)$ converges for considered values of η.

Recently, Adiyasuren and Batbold [1], derived a pair of Hilbert-type inequalities including a Hardy operator $\mathcal{A}:L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)$, defined by $(\mathcal{A}f)(x)=\frac{1}{x}\int_0^x f(t)dt$. Namely, assuming that $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a homogeneous function of degree $-(r+s), r,s>0$, they obtained inequalities

$$\int_0^\infty\int_0^\infty K(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathcal{A}f)(x)(\mathcal{A}g)(y)dxdy<pqk(1-s)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{1}$$

and

$$\left[\int_0^\infty y^{ps-1}\left(\int_0^\infty K(x,y)x^{r-\frac{1}{q}}(\mathcal{A}f)(x)dx\right)^p dy\right]^{\frac{1}{p}}<qk(1-s)\|f\|_{L^p(\mathbb{R}_+)}, \tag{2}$$

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where $\frac{1}{p}+\frac{1}{q}=1, p>1, 0<k(1-s)<\infty$, $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$, and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. The most interesting fact in connection with inequalities (1) and (2) is that the constant factors $pqk(1-s)$ and $qk(1-s)$ are the best possible (for more details, see [1]).

It should be noticed here that the Hardy operator $\mathcal{A}$ represents the arithmetic mean in integral case. In [2], Adiyasuren *et.al.* derived analogues of (1) and (2) with geometric and harmonic operators in both integral and discrete case.

On the other hand, considerable attention is given to the so-called half-discrete Hilbert-type inequalities, that is, to inequalities which include both integral and sum. Recently, Krnić *et.al.* [8], provided a unified treatment of half-discrete Hilbert-type inequalities with a homogeneous kernel and in the setting with non-conjugate exponents.

Suppose p and q are real parameters, such that

$$p>1, \quad q>1, \quad \frac{1}{p}+\frac{1}{q}\geqslant 1, \tag{3}$$

and let $p'=\frac{p}{p-1}$ and $q'=\frac{q}{q-1}$ respectively be their conjugate exponents, that is, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Further, define

$$\lambda=\frac{1}{p'}+\frac{1}{q'} \tag{4}$$

and observe that $0<\lambda\leq 1$ holds for all p and q as in (3). In particular, equality $\lambda=1$ holds in (4) if and only if $q=p'$, that is, only if p and q are mutually conjugate. Otherwise, we have $0<\lambda<1$, and such parameters p and q will be referred to as non-conjugate exponents.

In the above setting with non-conjugate exponents and with a homogeneous kernel K of degree $-s, s>0$, Krnić *et.al.* [8], have showed that the following triple of half-discrete Hilbert-type inequalities

$$\sum_{n=1}^{\infty}a_n\int_0^{\infty}K^{\lambda}(x,n)f(x)dx=\int_0^{\infty}f(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)a_n\right)dx<L\left[\int_0^{\infty}x^{\frac{p}{q'}(1-s)+p(\alpha_1-\alpha_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{\frac{q}{p'}(1-s)+q(\alpha_2-\alpha_1)}a_n^q\right]^{\frac{1}{q}}, \tag{5}$$

$$\left[\sum_{n=1}^{\infty}n^{\frac{q'}{p'}(s-1)+q'(\alpha_1-\alpha_2)}\left(\int_0^{\infty}K^{\lambda}(x,n)f(x)dx\right)^{q'}\right]^{\frac{1}{q'}}<L\left[\int_0^{\infty}x^{\frac{p}{q'}(1-s)+p(\alpha_1-\alpha_2)}f^p(x)dx\right]^{\frac{1}{p}} \tag{6}$$

and

$$\left[\int_0^{\infty}x^{\frac{p'}{q'}(s-1)+p'(\alpha_2-\alpha_1)}\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)a_n\right)^{p'}dx\right]^{\frac{1}{p'}}<L\left[\sum_{n=1}^{\infty}n^{\frac{q}{p'}(1-s)+q(\alpha_2-\alpha_1)}a_n^q\right]^{\frac{1}{q}}, \tag{7}$$

where $L=k^{\frac{1}{q'}}(q'\alpha_2)k^{\frac{1}{p'}}(2-s-p'\alpha_1)$, and α_1,α_2 are real parameters such that the function $K(x,y)y^{-q'\alpha_2}$ is decreasing on $\mathbb{R}_+$ for any $x\in\mathbb{R}_+$, holds for any non-negative measurable function $f:\mathbb{R}_+\to\mathbb{R}$ and a non-negative sequence $a=(a_n)_{n\in\mathbb{N}}$. Clearly, in the above inequalities all integrals and sums are assumed to be convergent, and the function and the sequence are not equal to zero. For some related half-discrete Hilbert-type inequalities, regarding some particular classes of kernels and weight functions, the reader is referred to the following references: [6,7,11,12,14,15].

It should be pointed out that the inequalities (5)–(7) are derived by virtue of a unified treatment of Hilbert-type inequalities, with a general kernel and weight functions, and with respect to measure spaces with positive σ-finite measures (for more details, see [3]).

The main objective of this paper is to derive half-discrete versions of inequalities (1) and (2), as well as the corresponding analogues with geometric and harmonic mean operators. Such inequalities may easily be derived by virtue of stated half-discrete inequalities (5)–(7), and several classical inequalities, such as the Hardy, the Knopp, and the Carleman inequality. A special emphasis is placed on establishing conditions under which such inequalities include the best possible constants on their right-hand sides.

The paper is divided into five sections as follows: After this Introduction, in Section 2 we introduce notations and list some important classical inequalities that will be necessary in establishing our main results. Further, in Section 3, we derive half-discrete analogues of inequalities (1) and (2), involving arithmetic, geometric and harmonic mean in the setting of non-conjugate exponents. After reduction to the case of conjugate exponents, in Section 4, we establish conditions for which derived inequalities include the best constants on their right-hand sides. As an application, in Section 5 we deal with certain half-discrete operators between Lebesgue spaces $L^p(\mathbb{R}_+)$ and l^p, arising from derived inequalities.

2. Notation and auxiliary results

In this article $L^p(\mathbb{R}_+), p\geqslant 1$, denotes the space of all Lebesgue measurable functions $f:\mathbb{R}_+\to\mathbb{R}$ such that $\|f\|_{L^p(\mathbb{R}_+)}=\left(\int_0^{\infty}|f(t)|^p dt\right)^{\frac{1}{p}}<\infty$. Similarly, $l^p, p\geqslant 1$, denotes the space of all real sequences $a=(a_n)_{n\in\mathbb{N}}$ such that $\|a\|_{l^p}=\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}<\infty$.