Applied Mathematics and Computation 231 (2014) 276–283

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An algorithm of polynomial order for computing the covering dimension of a finite space

D.N. Georgiou [a,*], A.C. Megaritis [b]

[a] Department of Mathematics, University of Patras, 265 04 Patras, Greece
[b] Optics and Optometry Department, Technological Educational Institute of Western Greece, Aigio, Greece

ARTICLE INFO

Keywords:
Algorithm of polynomial order
Covering dimension
Finite space
Incidence matrix

ABSTRACT

Finite topological spaces and the notion of dimension play an important role in digital spaces, computer graphics, image synthesis and image analysis (see, Herman, 1998 [9]; Khalimsky et al., 1990 [10]; Rosenfeld, 1979 [15]). In Georgiou and Megaritis (2011) [7] we gave an algorithm for computing the covering dimension of a finite space X using the notion of the incidence matrix of X. This algorithm has exponential order. In this paper we give a new algorithm of polynomial order for computing the covering dimension of a finite space.

1. Preliminaries

The class of finite topological spaces was first studied by P.A. Alexandroff in 1937 (see [1]). Many researchers are currently working on finite spaces (see, for example [2,3,5,6,8–13,15–18]).

In what follows we denote by $X = \{x_1, \dots, x_n\}$ a finite topological space of n elements and by $\mathbf{U}_i$ the smallest open set of X containing the point $x_i, i = 1, \dots, n$. Also, we denote by ω the first infinite cardinal.

The $n \times n$ matrix $T_X = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \in \mathbf{U}_j \\ 0, & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of X (see, for example [17]). We denote by $c_1, \dots, c_n$ the n columns of the matrix T_X and by $\mathbf{1}$ the $n \times 1$ matrix with all elements equal to one.

Let

$$c_i = \begin{pmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{pmatrix} \text{ and } c_j = \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{pmatrix}$$

be two $n \times 1$ matrices. We write $c_i \leqslant c_j$ if and only if $c_{ki} \leqslant c_{kj}$ for each $k = 1, \dots, n$. Also, by $\max c_i$ we denote the maximum of the set $\{c_{1i}, c_{2i}, \dots, c_{ni}\}$.

For the following notions see for example [4].

* Corresponding author.
E-mail addresses: georgiou@math.upatras.gr (D.N. Georgiou), megariti@master.math.upatras.gr (A.C. Megaritis).

0096-3003/$ - see front matter © 2014 Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.amc.2013.12.185

Let X be a space. A *cover* of X is a non-empty set of subsets of X, whose union is X. A cover c of X is said to be open (closed) if all elements of c is open (closed). A family r of subsets of X is said to be a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

Define the *order* of a family c of subsets of a space X as follows:

(a) $\mathrm{ord}(c) = -1$ if and only if $c = \{\emptyset\}$.
(b) $\mathrm{ord}(c) = k$, where $k \in \omega$, if and only if the intersection of any $k+2$ distinct elements of c is empty and there exist $k+1$ distinct elements of c, whose intersection is not empty.
(c) $\mathrm{ord}(c) = \infty$, if and only if for every $k \in \omega$ there exist k distinct elements of c, whose intersection is not empty.

The *covering dimension* of a space X (see, for example [4,14]), denoted by dim, is defined as follows: $\dim(X) \leqslant k$, where $k \in \{-1\} \cup \omega$ if and only if for every finite open cover c of the space X there exists a finite open cover r of X, refinement of c, with order less than or equal to k.

In [7] we gave an algorithm for computing the covering dimension dim of a finite topological space $X = \{x_1, \dots, x_n\}$ using the following propositions:

Proposition 1.1 (See Proposition 2.1 of [7]). *Let $X = \{x_1, \dots, x_n\}$ be a finite space. Then, $\dim(X) \leqslant k$, where $k \in \omega$ if and only if there exists an open cover $\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_m}\}$ of X such that $\mathrm{ord}(\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_m}\}) \leqslant k$.*

Proposition 1.2 (See Proposition 2.4 of [7]). *If $c_j = \mathbf{1}$ for some $j \in \{1, \dots, n\}$, then $\dim(X) = 0$.*

Proposition 1.3 (See Proposition 2.5 of [7]). *Let $c_{j_1}, \dots, c_{j_m}$ be m columns of the matrix T_X. Then, $c_{j_1} + \dots + c_{j_m} \geqslant \mathbf{1}$ if and only if the family $\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_m}\}$ is an open cover of X.*

Proposition 1.4 (See Proposition 2.6 of [7]). *Let $c_{j_1}, \dots, c_{j_m}$ be m columns of the matrix T_X and $k = \max(c_{j_1} + \dots + c_{j_m})$, that is k is the maximum element of the $n \times 1$ matrix $c_{j_1} + \dots + c_{j_m}$. Then,*

$$\mathrm{ord}(\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_m}\}) = k - 1.$$

Proposition 1.5 (See Proposition 2.9 of [7]). *Let $c_{j_1}, \dots, c_{j_m}$ be m columns of the matrix T_X such that $c_{j_1} + \dots + c_{j_m} \geqslant \mathbf{1}$. If $c_{r_1} + \dots + c_{r_q} \ngeqslant \mathbf{1}$ for every $q < m$ and $\{r_1, \dots, r_q\} \subseteq \{1, \dots, n\}$, then*

$$\dim(X) = \max(c_{j_1} + \dots + c_{j_m}) - 1.$$

An upper bound on the number of iterations of the algorithm is the number $2^n - 1$.

In this paper we give a new algorithm of polynomial order for computing the covering dimension of an arbitrary finite topological space. An upper bound on the number of iterations of this algorithm is the number $\frac{1}{2}n^2 + \frac{3}{2}n - 3$. In particular, for finite T_0-spaces, an upper bound on the number of iterations of this algorithm is the number $\frac{1}{2}n^2 - \frac{1}{2}n$.

2. A new algorithm for computing the covering dimension

Let $X = \{x_1, \dots, x_n\}$ be a finite space and T_X the $n \times n$ incidence matrix. In what follows we denote by $\mathcal{C}(X)$ the set of all subsets $\{x_{j_1}, \dots, x_{j_m}\}$ of X such that the family $\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_m}\}$ is an open cover of X. Also by $\leqslant$ we define a relation on the set $\mathcal{C}(X)$ as follows:

$$\{x_{j_1}, \dots, x_{j_{m_1}}\} \leqslant \{x_{j'_1}, \dots, x_{j'_{m_2}}\}$$

if and only if

$$\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_{m_1}}\} \subseteq \{\mathbf{U}_{j'_1}, \dots, \mathbf{U}_{j'_{m_2}}\}.$$

This relation is a preorder on the set $\mathcal{C}(X)$.

Definition 2.1. Every minimum element of $(\mathcal{C}(X), \leqslant)$ is called *minimal family*.

Remark 2.2.

(1) For every finite topological space X there exist minimal families on the set $\mathcal{C}(X)$ (see Proposition 2.4).
(2) If $\{x_{j_1}, \dots, x_{j_{m_1}}\}$ and $\{x_{j'_1}, \dots, x_{j'_{m_2}}\}$ are two minimal families, then $\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_{m_1}}\} = \{\mathbf{U}_{j'_1}, \dots, \mathbf{U}_{j'_{m_2}}\}$.