Applied Mathematics and Computation 231 (2014) 386–394

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability indices for randomly perturbed power systems

Humberto Verdejo [a,*], Wolfgang Kliemann [b], Luis Vargas [c]

[a] Department of Electrical Engineering, University of Santiago, Santiago, Chile
[b] Department of Mathematics, Iowa State University, IA, USA
[c] Department of Electrical Engineering, University of Chile, Santiago, Chile

ARTICLE INFO

Keywords:
Stochastic system
Stochastic stability
Lyapunov exponents
Moment stability
Electric power systems

ABSTRACT

This paper considers the stability of moments of stochastic systems, such as stability of the mean or mean-square stability. The exponential growth behavior of moments is compared to almost sure exponential growth via Lyapunov exponents. We develop a series of indices that are useful to describe system performance under random perturbations. The theory is applied to two examples, including an electric power system.

1. Introduction

Stability at operating points is one of the key requirements of engineering systems. As long as the system is given by time-invariant dynamics, linearization at the operating point gives local stability information that can be extended through incorporating some nonlinear features, e.g., via the use of normal forms, see [1]. If the system under consideration has time-varying dynamics, the usual modal approach fails since for these systems eigenvalues do not describe the stability behavior of the linearized system. Therefore, one has to approach (exponential) stability directly via the Lyapunov exponents of the system at the operating point.

An important class of systems with time varying dynamics are those systems that are subject to sustained random perturbations, such as load behavior, environmental effects, or intermittent generation in power systems. The interaction between system dynamics and perturbation falls into two groups: (i) the random noise changes the operating point of the system, or (ii) the equilibrium point persists under all perturbations. We have developed performance indices for case (i) in [2], and analyzed one specific approach in case (ii) in [3] using almost sure Lyapunov exponents. This paper develops several performance indices for case (ii), analyzes their relationships, and compares the results for several examples. The key idea is the look at the sample (exponential) growth rates for trajectories and at the growth rates of moments of the trajectories, such as the stability of the mean, or mean square stability involving the second moment. Both points of view result in potentially useful performance criteria for power systems.

2. Mathematical background

2.1. The system model

We start from a nonlinear differential equation $\dot{y}(t) = f(y(t), \xi(t,\omega))$ in $\mathbb{R}^d$ with sustained random perturbation $\xi(t,\omega)$. In order to analyze optimal parameter settings for stability at an operating point, we linearize the system equations at the equilibrium point y^*. Linearization (with respect to y) at the equilibrium results in the system

* Corresponding author.
E-mail address: humberto.verdejo@usach.cl (H. Verdejo).

0096-3003
http://dx.doi.org/10.1016/j.amc.2014.01.024

$$\dot{x}(t) = A(\xi(t,\omega))x(t) \quad \text{in } \mathbb{R}^d \tag{1}$$

where $A(\xi(t,\omega))$ is the Jacobian of $f(y(t),\xi(t,\omega))$ at y^*. We denote by $\varphi(t,x,\omega)$ the trajectories of (1) with initial value $\varphi(0,x,\omega) = x \in \mathbb{R}^d$. We think of a given probability space $(\Omega, \mathfrak{F}, P)$ under the usual conditions on which the Wiener process in (2) is defined. We use the notation $\omega \in \Omega$, and all expectations $\mathbb{E}(\cdot)$ are with respect to the given probability measure P.

The random perturbation can be considered as white noise, leading to a stochastic differential equation for (1), or as a colored, bounded noise. In this paper we discuss the latter situation since macroscopic perturbations in engineering systems generally are non-white; but a similar theory also holds for the white noise case, see [4] for the basics. We start from a background noise η, given by a stochastic differential equation on a compact C^∞-manifold M

$$d\eta = X_0(\eta)dt + \sum_{i=1}^{r} X_i(\eta) \circ dW_i \text{ on } M \tag{2}$$

where the vector fields $X_0, \ldots, X_r$ are C^∞, and "$\circ$" denotes the Stratonovic stochastic differential. We assume that (2) has a unique stationary, ergodic solution $\eta^*(t,\omega)$ which is guaranteed by the condition (compare [5])

$$\dim \mathcal{LA}\{X_1, \ldots, X_r\}(\theta) = \dim M \text{ for all } \theta \in M. \tag{3}$$

Here $\mathcal{LA}\{\cdot\}$ denotes the Lie algebra generated by a set of vector fields. The background noise $\eta^*(t,\omega)$ is mapped via a surjective smooth function $f : M \to U \subset \mathbb{R}^m$, $f(\eta) = \xi$, into the system perturbation $\xi(t,\omega)$. This setup allows great flexibility when modeling the statistics of the system noise.

2.2. Lyapunov exponents

Exponential stability of the system (1) is described by Lyapunov exponents; in [6] we gave an overview of the almost sure theory, with applications to power systems. Here we extend the analysis to moment Lyapunov exponents. The individual Lyapunov exponents of the trajectories $\varphi(t,x,\omega)$ of (1) are given as

$$\lambda(x,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log|\varphi(t,x,\omega)|, \tag{4}$$

and for $p \in \mathbb{R}$ the Lyapunov exponent of the pth moment is given by

$$g(p,x) = \limsup_{t\to\infty} \frac{1}{t} \log \ \mathbb{E}|\varphi(t,x,\omega)|^p. \tag{5}$$

This includes for $p = 1$ the exponential growth behavior of the mean, and for $p = 2$ the exponential mean square stability of the system. We again need the projection of the linear system onto the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$:

$$\dot{s}(t) = h(\xi(t,\omega), s(t)), \ \ h(\xi,s) = (A(\xi) - q(\xi,s))s, \ \ q(\xi,s) = s^T A(\xi)s, \tag{6}$$

where "$\cdot^T$" denotes the transpose. via identification of s and $-s$ Eq. (6) can be considered on the projective space $\mathbb{P}^{d-1}$. The Lyapunov exponents of all system states $x \in \mathbb{R}^d \setminus \{0\}$ can be analyzed together if the perturbation affects all states. This is expressed in the condition

$$\dim \mathcal{LA}\left\{\left(X_o, h, \frac{\partial}{\partial t}\right), (X_1, 0, 0), \ldots, (X_r, 0, 0)\right\}(\theta, s, t) = \dim M + d \tag{7}$$

for all $(\theta, s, t) \in M \times \mathbb{S}^{d-1} \times \mathbb{R}$. Another approach to condition (7) is as follows: Let $\mathcal{I}$ be the ideal in $\mathcal{LA}\{X_o + h, X_1, \ldots, X_r\}$ generated by $\{X_1, \ldots, X_r\}$. Then, by [5], Condition (7) is equivalent to $\dim \mathcal{I}(\theta, s) = \dim M + d - 1$. This condition, which is needed for the analysis of moment Lyapunov exponents, is slightly stronger than Condition 7 in [6], but it is generally satisfied for systems that appear in applications, compare, e.g., [5] or [10].

Theorem 2.1. *Consider the stochastic system (1) under the conditions (3)* and *(7). Then*

1. *the moment Lyapunov exponents exist as a limit and they are independent of $x \in \mathbb{R}^d \setminus \{0\}$, i.e., $g(p) \equiv g(p,x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}|\varphi(t,x,\omega)|^p$ for all $p \in \mathbb{R}$,*
2. *the trajectory-wise Lyapunov exponents are a.s. constant and independent of $x \in \mathbb{R}^d \setminus \{0\}$, i.e., $\lambda \equiv \lambda(x,\omega) = \lim_{t\to\infty} \frac{1}{t} \log|\varphi(t,x,\omega)|$.*

The proof of Theorem 2.1 is given in [7], Theorem 1 for the first part, and in [10], Theorem 4.1 for the second part upon noticing that Conditions (3) and (7) together imply Conditions (A) and (C) in [10]. With the results from Theorem 2.1 it was shown by Arnold in [7] that the a.s. Lyapunov exponent is the derivative of the moment Lyapunov exponent function at 0:

Corollary 2.2. *Consider the stochastic system (1) under the conditions (3)* and *(7). Then the function $g(p)$ is analytic on $\mathbb{R}$, convex, and satisfies $g(0) = 0$ and $g'(0) = \lambda$.*