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Instabilities and propagation properties in a fourth-order reaction–diffusion equation

Samir A. Shamseldeen

Mathematics and Engineering Physics Dept., Faculty of Engineering, Mansoura University, Mansoura 35516, Egypt

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ABSTRACT

In this paper we investigate the instability and the propagation properties of a class of reaction–diffusion equations of fourth order. Two examples are introduced, the extended Fisher Kolmogorov equation (EFK), and the Swift–Hohenberg equation (SH). Both have been studied before by related methods (see for example, Peletier and Rottschafer, 2004 [19]; Van Saarloos, 2003 [24]) but the analysis here will support the introduced linear mechanism in front selection. These two equations support a patterned front solutions, and the double eigenvalue mechanism is used to provide evidence for that and to determine a minimal front speed.

1. Introduction

Whenever the spatial spread of a population or chemical species is of importance, *reaction–diffusion equations* are used. For spatial spread, reaction–diffusion models have successfully been used in epidemic problems, pattern formation in different biological and ecological systems and in signal transport. Good overviews are given in Britton [5] and Grindrod [11]. From a theoretical point of view, one may distinguish two types of reaction–diffusion structures: (i) global structures resulting from intrinsic symmetry-breaking instabilities, e.g., Turing structures [21], and (ii) localized structures associated with fronts, i.e., steep spatial changes of concentration or densities which correspond to transitions between two states with fast kinetics, e.g., traveling waves [5]. In many natural phenomena we encounter propagating fronts separating different phases. Propagating fronts play an important role in the spread of epidemics, in population dynamics, or the propagation of flames and chemical reactions. Therefore, reaction–diffusion equations have become a prototype for describing propagating front behavior, form chemical waves to biological population.

1.1. Traveling waves and front propagation

A traveling wave is a wave which travels at constant speed without change in shape. If $u(\mathbf{x}, t)$ represents a traveling wave, the shape of u will be the same for all time and the speed of propagation of this shape is a constant. If we look at this wave in a traveling frame moving at the same speed it will appear stationary [16]. One of the most important properties of nonlinear parabolic systems is their ability to support traveling wave solutions. Unlike the linear wave equation, for example, which is hyperbolic and propagates any wave profile with a specific speed, reaction–diffusion equations may allow various wave profiles to propagate, each one with its own characteristic speed [11].

Traveling wave solution can be written in the form $u(\mathbf{x}, t) = V(\mathbf{z}) = V(\mathbf{x} - \mathbf{c}t)$ for some velocity $\mathbf{c}$. Plane wave is a class of traveling waves with $V(\mathbf{z}) = U(\mathbf{z} \cdot \mathbf{s})$ for some vector $\mathbf{s}$) (i.e., $u = U(\mathbf{z} \cdot \mathbf{s} - ct)$), c a scalar). This class of waves, plane waves, is

E-mail address: samir@mans.edu.eg

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categorized in one dimension as [9]: (1) wave trains (U periodic), (2) fronts ($U(-\infty)$ and $U(\infty)$ exist and are unequal) and (3) pulses ($U(\pm\infty)$ exist and are equal; U not constant). There are other forms in two dimension ($\mathbf{x} = (x, y),\ x = r\cos\theta,\ y = r\sin\theta$), such as: Target patterns ($u(\mathbf{x}, t) = U(r, t),\ U$ periodic in t), and Rotating spiral patterns ($u(\mathbf{x}, t) = U(r, \theta - ct),\ U$ periodic in second argument).

The propagation of a front into an unstable state is a problem that emerges in many branches of the natural sciences. These fronts may be classified as: (1) *Uniformly translating fronts*, which are in the form $u(z) = u(x - ct)$, where c is the front speed. In this class of fronts invasion could be either monotonic or oscillatory (see Fig. 1(a) and (b), which represent possible solutions of Fisher's equation (1)). (2) *Pattern forming fronts*, a front that generates a nontrivial pattern behind the wavefront. The front has a finite speed while the pattern is often stationary (see Fig. 1(c), represent possible solution of SH equation [6]). Thus these pattern fronts are typically not in the form $u(x - ct)$, and instead they are spatially and temporally periodic: they are of the type $u(z, t) = u(x - ct, t)$, with $u(z, t)$ periodic in t with period $T, u(z, t) = u(z, t + T)$, thus in our analysis the perturbations are assumed in that form as we will see later.

1.2. Front selection

The prototypical model for reaction–diffusion systems is the Fisher-type nonlinear diffusion equation (scalar monostable), which we use here to illustrate some general principles:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{1}$$

where $u > 0$ may be interpreted as a population density, $F(0) = F(1) = 0$. This equation was introduced in 1937 by Fisher [10], with $F(u) = u(1 - u)$. At the same time by Kolmogorov, together with Petrovskii and Piskunov [13] (hereafter (1) referred to as FKPP). In their work of 1937, Kolmogorov et al. proved the existence of front solutions $u = U(x - ct)$, characterized by their velocity, c, such that

$$c \geqslant c_0 = 2\sqrt{F'(0)} \tag{2}$$

and this result is obtained by a linearization about $u = 0$. Moreover, under some assumption on F, they proved that the FKPP-equation, Eq. (1), with a sufficiently decaying initial data has solutions with speed c_0. For more general monostable equations, it was shown rigorously by Aronson and Weinberger [2], for a sufficiently localized initial condition the solutions of (1) evolve into fronts with a minimal allowed speed c_{min}, such that

$$2\sqrt{F'(0)} \leqslant c_{min} \leqslant 2\sup_u \sqrt{F(u)/u}, \tag{3}$$

thus the propagating speed is either equal to or larger than c_0. Also, they showed that a monotonic traveling wave exists for all speeds $c \geqslant c_{min}$, and none for $c < c_{min}$. Therefore, from these results two selection mechanisms appeared: a *linear* and *nonlinear* selection of the propagation speed. In a linear selection mechanism the front dynamics can be understood by linear analysis since it is essentially determined by linearization near the unstable steady state ($u = 0$ in case of FKPP equation), so the front is pulled by its leading edge (see Fig. 1(a)), and in this case the selected front is called *pulled* front. However, for the selected fronts with speeds larger than the linear front speed, the details of the nonlinearity of the reaction term, $F(u)$, are important. In this case, the front dynamics are referred to as *pushed*, meaning that the front is pushed by its (nonlinear) interior, and a nonlinear analysis is required to determine the front speed. A nonlinear selection principle has been proposed to that aim (see [24]). Fisher's equation has been studied extensively, considering the traveling wave existence

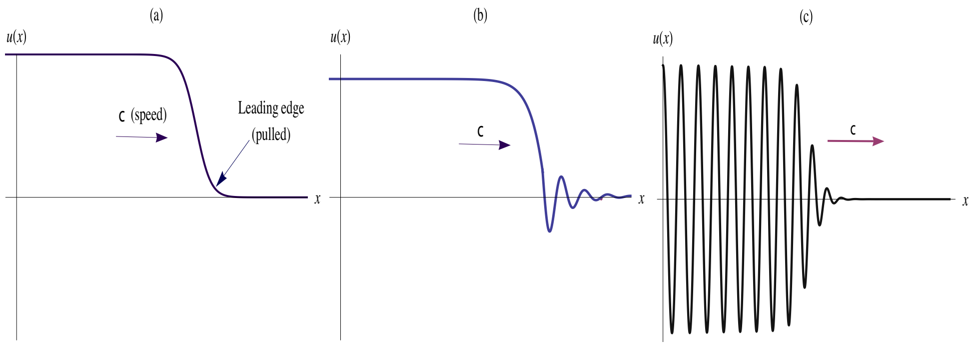

Fig. 1. Schematic representation of some front types, all moving to the right with speed c. (a) Monotonic uniform translating front, represents solution of Fisher's equation (1) when speed $c > 2$. (b) Front invading the unstable state in oscillatory manner, represents solution of Fisher's equation (1) when speed $c < 2$. (c) Pattern forming front, a front moving to the right leaving a pattern behind. There are possible states behind the front, such as limit cycles, stationary patterns, oscillatory patterns, and spatio-temporal patterns.