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# A simple class of fractal transforms for hyperspectral images

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ABSTRACT

A complete metric space of function-valued mappings appropriate for the representation of hyperspectral images is introduced. A class of fractal transforms is then formulated on this space. Under certain conditions, the fractal transform $T$ can be contractive, implying the existence of a unique fixed point. We then formulate a simple class of block transforms for the fractal coding of digital hyperspectral images, and illustrate with an example.


## 1. Introduction

In [22], we examined some basic self-similarity properties of hyperspectral (HS) images, considering them as *function-valued mappings* of a base (or pixel) space $X$ to a suitable (spectral) function space. At each location or pixel $x \in X$, the hyperspectral image mapping $u(x)$ is a *function* that is supported on a suitable domain of definition $Y$. In practical applications, of course, HS images are digitized: Both the base space $X$ and spectral domain $Y$ are discretized so that $u(x)$ is a vector.

Earlier studies of greyscale images [1,5] have shown that most subblocks of natural images are well approximated (using various forms of affine greyscale mappings) by a number of other subblocks of the image. Such *image self-similarity* is responsible, at least in part, for the effectiveness of various non-local image processing schemes, including nonlocal-means denoising [6], fractal image coding [12,18] and a variety of other methods devoted to image enhancement, e.g., [8–11,14]. The study in [22] shows that HS images are also quite self-similar, in the sense that "data cubes", namely, $M$-channel vectors supported over $n \times n$-pixel subblocks of the HS image are well approximated by a number of other data cubes of the image. Moreover, the spectral functions over individual pixels demonstrate a remarkable degree of correlation with each other, not only locally but over the entire image. This suggests that various nonlocal image processing schemes which rely on self-similarity should be quite effective for HS images.

In Section 3 of this paper, we provide the mathematical formalism for a particular class of *affine fractal transforms* on the space of function-valued HS images and show that under certain conditions, a fractal transform $T$ can be contractive. From Banach's Fixed Point Theorem, this implies the existence of a fixed point HS image $\bar{u}$ such that $T\bar{u} = \bar{u}$. This leads to the *inverse problem* of fractal image coding, namely, given an HS image $u$, find a fractal transform $T$ with fixed point $\bar{u}$ that approximates $u$ to a sufficient degree of accuracy. As in the case of fractal coding of greyscale images, this problem can be solved by means of *collage coding*, i.e., find a fractal transform $T$ that maps the HS image $u$ as close to itself as possible.

One of the original motivations for fractal image coding was *image compression* [4,12,18]. As in the case of standard transform coding of images, it was found that much less computer memory was required to store the parameters defining the block-based fractal transform $T$ of an image $u$. Moreover, the fixed-point approximation $\bar{u}$ to $u$ can be constructed by iteration

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of the transform $T$. Fractal image coding has been shown to be effective in performing a number of other image processing tasks, for example, denoising [15] and super-resolution [19].

In Section 5, we examine in more detail a block fractal coding scheme briefly introduced in [22], deriving sufficient conditions for contractivity of the associated fractal transform $T$. We also present the results of some computations on a hyperspectral image. However, it is not the purpose of the present paper to investigate the compression capabilities of this fractal coding scheme nor to compare it with other compression schemes.

Acknowledging the tremendous amount of work that has been done on hyperspectral images, e.g., [7,21], we mention that our work is intended to complement the well-established notion that hyperspectral images generally exhibit a high degree of correlation which can be exploited for the purposes of image enhancement.

## 2. A complete metric space $(Y, d_Y)$ of hyperspectral images

We consider hyperspectral images as function-valued mappings of a *base space* $X$ to an appropriate space of *spectral functions* $\mathcal{F}$, along the lines established in [22,20]. In this paper, the ingredients of our formalism are as follows:

- **The base space** $X$**:** The compact support of the hyperspectral images, with metric $d_X$. For convenience, $X = [0,1]^n$, where $n = 1, 2$ or 3.
- **The range or spectral space** $\mathcal{F}$**:** The space $L^2(\mathbf{R}_s)$ of square-integrable functions supported on a compact set $\mathbf{R}_s \subset \mathbf{R}_+$, where $\mathbf{R}_+ = \{y \in \mathbf{R} | y \geqslant 0\}$. $L^2(\mathbf{R}_s)$ is a Hilbert space with the standard definition of the inner product, i.e.,

$$\langle f, g\rangle = \int_{\mathbf{R}_s} f(t)g(t)\,dt, \quad \forall f, g \in L^2(\mathbf{R}_s). \tag{1}$$

This inner product defines a norm on $L^2(\mathbf{R}_s)$, to be denoted as $\|\cdot\|_{L^2(\mathbf{R}_s)}$.

We now let $Y$ denote the set of all function-valued mappings from $X$ to $L^2(\mathbf{R}_s)$. Given a hyperspectral image $u \in Y$, its value $u(x)$ at a particular location $x \in X$ will be a function – more precisely, an element of the space $L^2(\mathbf{R}_s)$. Following the same prescription as in [20], the norm $\|\cdot\|_{L^2(\mathbf{R}_s)}$ arising from Eq. (1) may be used to define a norm $\|\cdot\|_Y$ on $Y$ which, in turn, defines a metric $d_Y$ on $Y$. The distance between two hyperspectral images $u, v \in Y$ will then be defined as

$$d_Y(u, v) = \left[\int_X \|u(x) - v(x)\|^2_{L^2(\mathbf{R}_s)}\,dx\right]^{1/2}. \tag{2}$$

It is straightforward to show that the metric space $(Y, d_Y)$ of hyperspectral images is complete. In fact, $Y$ is trivially a Hilbert space.

## 3. A class of fractal transforms on $(Y, d_Y)$

We now list the ingredients for a class of fractal transforms on the space of HS images introduced above. For simplicity (especially as far as notation is concerned), we assume that our HS images are "one-dimensional," i.e., $X = [0,1]$. The extension to $[0,1]^n$, in particular, $n = 2$, is straightforward.

1. A set of $N$ one-to-one, affine contraction mappings $w_i : X \to X$, $w_i(x) = s_i x + a_i$, $x \in X$, with the condition that $\cup_{i=1}^N w_i(X) = X$. In other words, the contracted copies, or "tiles" of $X$, $w_i(X)$, cover $X$.
2. Associated with each map $w_i$ are the following:
   (a) A scalar $\alpha_i \in \mathbf{R}$ and
   (b) A function $\beta_i : \mathbf{R}_s \to \mathbf{R}_+$, $\beta_i \in L^2(\mathbf{R}_s)$.

The action of the fractal transform $T : Y \to Y$ defined by the above is as follows: For a $u \in Y$ and any $x \in X$,

$$v(x) = (Tu)(x) = {\sum_{i=1}^N}' [\alpha_i(u(w_i^{-1}(x))) + \beta_i]. \tag{3}$$

The prime on the summation signifies that we sum over only those $i \in \{1, 2, \ldots, N\}$ for which the preimage $w_i^{-1}(x)$ exists, i.e., those $i$ for which $x \in w_i(X)$.

The above formulation represents a generalization of the standard fractal transform for greyscale images. The "value" of the HS image $v(x) = (Tu)(x)$ at a point $x \in X$ is a *spectral function*, i.e., $v(x) \in L^2(\mathbf{R}_s)$. Furthermore, the values of $v(x)$ at $t \in \mathbf{R}_s$ are given by

$$v(x; t) = (Tu)(x; t) = {\sum_{i=1}^N}' [\alpha_i u(w_i^{-1}(x); t) + \beta_i(t)]. \tag{4}$$

The function $\beta_i(t)$ replaces the traditional constant $\beta_i$ employed in standard fractal transforms for (single-valued) images [12,18].