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Another look at some new Cauchy–Schwarz type inner product inequalities

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ABSTRACT

In this paper we take a refreshing look at a Cauchy–Schwarz type inner product inequalities. We also provide a Cauchy–Schwarz based proof and a refinement of Buzano's inequality in inner product spaces. Some elementary applications such as trace inequalities for unitary matrices and discrete or integral inequalities are given.

1. Introduction

The theory of inner product inequalities plays a central role in many branches of mathematics like Linear Operators, Partial Differential Equations, Approximation & Optimization Theory, Information Theory or Statistics and many other fields. Such inequalities were the pioneering work of mathematicians like Cauchy, Minkovski, Holder, Hilbert, Hardy, Kantorovich and many others.

On the other hand, it is worth mentioning the contributions of Bellman, Boas, van der Corput, Ostrowski, Selberg, Enflo and Bombieri who have applied successfully in obtaining applications for Fourier and Mellin transforms, oscillatory integrals, approximation of polynomials or large sieve. All these results as well as their applications are covered in the monograph [4].

In [2], Buzano gave the following extension of the celebrated Cauchy–Schwarz's inequality in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$,

$$|\langle a,c\rangle\langle c,b\rangle| \leqslant \frac{1}{2}(||a||||b|| + |\langle a,b\rangle|)\cdot ||c||^2 \tag{1}$$

for all $a,b,c\in H$.

Clearly, when $a=b$, the above inequality becomes the celebrated Cauchy–Schwarz's inequality,

$$|\langle a,c\rangle|^2 \leqslant ||a||^2||c||^2 \tag{2}$$

for all $a,c\in H$.

As pointed out in [5], the original proof of Buzano is a little bit complicated. Buzano's inequality also mentioned in [8] as an interesting generalization of the Cauchy–Schwarz's inequality and its proof requires some facts about orthogonal

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decomposition of a complete inner product space. Moreover, Fuji and Kubo [5] gave a simple proof by using the orthogonal projection on a subspace of an inner product space H and Cauchy–Schwarz's inequality.

The equality case in (1) holds if

$$c = \begin{cases} \alpha\left(\frac{a}{||a||} + \frac{\langle a,b\rangle}{|\langle a,b\rangle|}\cdot\frac{b}{||b||}\right), & \text{when } \langle a,b\rangle \neq 0, \\ \alpha\left(\frac{a}{||a||} + \beta\frac{b}{||b||}\right), & \text{when } \langle a,b\rangle = 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A detailed proof of Buzano's inequality can also be found in [15], pp. 60–61.

For real inner product spaces, Richard [14] obtained the following stronger inequality

$$\left|\langle a,c\rangle\langle c,b\rangle - \frac{1}{2}\langle a,b\rangle||c||^2\right| \leqslant \frac{1}{2}||a||||b||||c||^2, \tag{3}$$

$a, b, c \in H$.

Dragomir [4] showed that the above inequality is true with coefficients $\frac{1}{|\alpha|}$ instead of $\frac{1}{2}$, where α is a non-zero number with $|1-\alpha| = 1$. Also, Dragomir [4] obtained a refinement of Buzano's inequality,

$$|\langle a,c\rangle\langle c,b\rangle| \leqslant |\langle a,c\rangle\langle c,b\rangle - \frac{1}{2}\langle a,b\rangle||c||^2| + \frac{1}{2}\langle a,b\rangle||c||^2 \leqslant \frac{1}{2}(||a||||b|| + |\langle a,b\rangle|)||c||^2, \tag{4}$$

$a, b, c \in H$.

In [6], Gavrea generalized Buzano's inequality to nonegative real exponents. In fact, he proved that

$$|\langle a,x\rangle|^{\alpha} \cdot |\langle x,b\rangle|^{\beta} \leqslant t_1^{\frac{\alpha}{2}} t_2^{\frac{\beta}{2}} ||x||^{\alpha+\beta},$$

where t_1 and t_2 are expressed in terms of the constants α and β and vectors a, b.

2. A Cauchy–Schwarz setting to prove Buzano's inequality

In this section we prove Buzano's inequality using a Krein type inequality which is deduced only from Cauchy–Schwarz's inequality. As mentioned in [9], for any given two vectors $a, b \in \mathbb{C}^n - \{0\}$, one can define

$$\cos\Phi_{ab} = \frac{\mathrm{Re}\langle a,b\rangle}{||a||||b||}$$

and

$$\cos\Psi_{ab} = \frac{|\langle a,b\rangle|}{||a||||b||}.$$

In 1969, Krein proved the following inequality,

$$\Phi_{ac} \leqslant \Phi_{ab} + \Phi_{bc}$$

for any $a, b, c \in \mathbb{C}^n$.

A proof of the above result can be found in [7,9]. Moreover, the main idea from the proof of Krein's inequality is hidden in the following Krein-type inequality,

Theorem 2.1. *For any vectors $a, b, c \in (H, \langle\cdot,\cdot\rangle)$ we have the following inequality*

$$||a||^2|\langle b,c\rangle|^2 + ||b||^2|\langle c,a\rangle|^2 + ||c||^2|\langle a,b\rangle|^2 \leqslant ||a||^2||b||^2||c||^2 + 2|\langle a,b\rangle\langle b,c\rangle\langle c,a\rangle|.$$

First proof. We will give a proof using only Cauchy–Schwarz's inequality. Without loss of generality, we assume $||c|| \neq 0$, otherwise the inequality is trivial. Now, for any complex scalar λ, by the Cauchy–Schwarz's inequality, we have

$$||a-\lambda b||^2||c||^2 \geqslant |\langle a-\lambda b, c\rangle|^2$$

for all $a, b, c \in H$. By expanding, this inequality is successively equivalent to

$$(||a||^2 - 2\lambda|\langle a,b\rangle| + \lambda^2||b||^2)||c||^2 \geqslant |\langle c,a\rangle|^2 - 2\lambda|\langle c,a\rangle||\langle b,c\rangle| + \lambda^2|\langle c,a\rangle|^2,$$

$$(||b||^2||c||^2 - |\langle b,c\rangle|^2)\lambda^2 + 2\lambda(|\langle a,b\rangle|||c||^2 - |\langle c,a\rangle||\langle b,c\rangle|) + ||a||^2||c||^2 - |\langle c,a\rangle|^2 \geqslant 0$$

for all complex scalars λ. The last inequality can be viewed as a quadratic polynomial in λ with its dominant coefficient non-negative (by Cauchy–Schwarz's inequality), and thus, the discriminant Δ must be negative,

$$\Delta_{\lambda} = 4(|\langle a,b\rangle|||c||^2 - |\langle c,a\rangle||\langle b,c\rangle|)^2 - 4(||b||^2||c||^2 - |\langle b,c\rangle|^2)(||a||^2||c||^2 - |\langle c,a\rangle|^2) \leqslant 0,$$

which is equivalent to