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Existence of bounded solutions of a class of neutral systems of functional differential equations

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ABSTRACT

Some results on the existence of bounded solutions together with their first derivatives of a class of neutral systems of functional differential equations with complicated deviations, which extend and unify numerous results in the literature, are proved.

1. Introduction and preliminaries

Special cases of the following system of functional differential equations, which is partially solved with respect to the first derivatives of dependent variables,

$$x'(t+1) = Ax'(t) + \Phi(t, x(t), x(f_1(t, x(t))), x'(f_2(t, x(t)))), \tag{1}$$

where $t \in \mathbb{R}_+ = [0, \infty)$, $\Phi : \mathbb{R}_+ \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$, $f_i : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$, $i = 1, 2$, have attracted some attention among the experts in the research field (see, for example, [1,14,19,20,22,23,44,47,48]). For some other results on systems/equations not solved with respect to the highest-order derivatives, see, for example, [3–13,16–18,21,24,38,40,46,49]. Based on the idea of iterations of some iterative processes (see, for example, [2,15,25–37,42]) in [38–41,43–47], we proposed the investigation of various types of systems/equations with continuous arguments, whose deviations of an argument depend on an unknown function which depend also of the function and so on, so called, *iterated deviations*.

Motivated by the line of investigations in the papers [4,13,16,17,21,22,38,39,44,46–48], here we investigate the existence of bounded C^1 solutions of the next system of functional differential equations

$$x'(t+1) = Ax'(t) + \Phi(t, x(v_1^{(1)}(t)), \ldots, x(v_1^{(k)}(t)), x'(u_1^{(1)}(t)), \ldots, x'(u_1^{(l)}(t))), \tag{2}$$

on $\mathbb{R}_+$, where

$$v_r^{(j)}(t) = \varphi_{jr}(t, x(\varphi_{j\,r+1}(t, \ldots x(\varphi_{jm_j}(t, x(t))) \ldots))),$$
$$u_p^{(i)}(t) = \psi_{ip}(t, x(\psi_{i\,p+1}(t, \ldots x(\psi_{i\mu_i}(t, x(t))) \ldots))),$$

$j = \overline{1,k}$, $r = \overline{1,m_j}$, $i = \overline{1,l}$, $p = \overline{1,\mu_i}$, $\Phi : \mathbb{R}_+ \times (\mathbb{R}^N)^{k+l} \to \mathbb{R}^N$, $\varphi_{jr}, \psi_{ip} : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$, $j = \overline{1,k}$, $r = \overline{1,m_j}$, $i = \overline{1,l}$, $p = \overline{1,\mu_i}$, A is a nonsingular matrix, extending and unifying numerous results in the literature.

We use also the following convention

$$v_{m_j+1}^{(j)}(t) = u_{\mu_i+1}^{(i)}(t) = t, \quad j = \overline{1,k}, \; i = \overline{1,l}. \tag{3}$$

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As usual, by $C(\mathbb{R}_+)$ we denote the space of continuous vector functions on $\mathbb{R}_+$, while by $C^1(\mathbb{R}_+)$ the space of all continuously differentiable vector functions on $\mathbb{R}_+$. The subspace of $C^1(\mathbb{R}_+)$ consisting of all bounded vector functions together with their first derivatives on $\mathbb{R}_+$ is denoted by $BC^1(\mathbb{R}_+)$. The norm on $BC^1(\mathbb{R}_+)$ is

$$\|x\|_{BC^1(\mathbb{R}_+)} = \max\{\|x\|_\infty, \|x'\|_\infty\} = \max\left\{\sup_{t\in\mathbb{R}_+}|x(t)|, \sup_{t\in\mathbb{R}_+}|x'(t)|\right\},$$

where for $y \in \mathbb{R}^N$, $|y|$ denotes a norm on $\mathbb{R}^N$.

The following folklore lemma, which can be found, for example, in [46], will be frequently applied in the proofs of our main results.

Lemma 1. *Assume that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of nonnegative numbers, and that sequence $(x_n)_{n\in\mathbb{N}}$ satisfies the inequality*

$$x_n \leqslant a_n + b_n x_{n+1}, \qquad n \in \mathbb{N}.$$

Then

$$x_1 \leqslant \sum_{j=1}^{k-1} a_j \prod_{i=1}^{j-1} b_i + x_k \prod_{i=1}^{k-1} b_i, \quad k \in \mathbb{N}.$$

2. Main results

First, we give a list of some conditions which will be used in the formulations of the main results in this paper.

(a) Vector function $\Phi(t, x_1, \ldots, x_{k+l})$ is continuous for $t \in \mathbb{R}_+$, $x_j \in \mathbb{R}^N$, $j = \overline{1, k+l}$,

$$\Phi(t, 0, \ldots, 0) \equiv 0, \tag{4}$$

$$|\Phi(t, x'_1, \ldots, x'_{k+l}) - \Phi(s, x''_1, \ldots, x''_{k+l})| \leqslant \gamma_0(t,s)|t-s| + \sum_{j=1}^{k+l} \gamma_j(t,s)|x'_j - x''_j|, \tag{5}$$

where $\gamma_j(t,s)$, $j = \overline{0, k+l}$ are continuous and nonnegative functions for $t, s \in \mathbb{R}_+$, and $x'_j, x''_j \in \mathbb{R}^N$, $j = \overline{1, k+l}$;

(b) $\varphi_{jr}(t,x)$, $j = \overline{1,k}$, $r = \overline{1, m_j}$, and $\psi_{ip}(t,x)$, $i = \overline{1,l}$, $p = \overline{1, \mu_i}$, are continuous and nonnegative functions for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$, and

$$|\varphi_{jr}(t,x) - \varphi_{jr}(s,y)| \leqslant \lambda_{jr}^{(1)}|t-s| + \lambda_{jr}^{(2)}|x-y|, \quad j = \overline{1,k}, \ r = \overline{1, m_j}, \tag{6}$$

$$|\psi_{ip}(t,x) - \psi_{ip}(s,y)| \leqslant \lambda_{ip}^{(3)}|t-s| + \lambda_{ip}^{(4)}|x-y|, \quad i = \overline{1,l}, \ p = \overline{1, \mu_i}, \tag{7}$$

for every $t, s \in \mathbb{R}_+$, and $x, y \in \mathbb{R}^N$, and for some positive constants $\lambda_{jr}^{(1)}$, $\lambda_{jr}^{(2)}$, $j = \overline{1,k}$, $r = \overline{1, m_j}$, $\lambda_{ip}^{(3)}$, $\lambda_{ip}^{(4)}$, $i = \overline{1,l}$, $p = \overline{1, \mu_i}$;

(c) for every $j = \overline{0, k+l}$, the series

$$\Gamma_j(t,s) = \sum_{i=0}^{\infty} |A^{-1}|^{i+1} \gamma_j(t+i, s+i) \quad \text{and} \quad G_j(t) = \sum_{i=0}^{\infty} |A^{-1}|^{i+1} \int_t^{\infty} \gamma_j(\tau+i, \tau+i) d\tau,$$

converge uniformly for $t, s \in \mathbb{R}_+$, and for some $\delta \in (0,1)$, satisfy the condition

$$\max\left\{\sup_{t,s\in\mathbb{R}_+} \sum_{j=0}^{k+l} \Gamma_j(t,s), \ \sup_{t\in\mathbb{R}_+} \sum_{j=0}^{k+l} G_j(t)\right\} \leqslant \delta. \tag{8}$$

Theorem 1. *Suppose that conditions (a)–(c) hold. Then for any $BC^1(\mathbb{R}_+)$ solution of system (2), such that*

$$\lim_{t\to+\infty} |x(t+1) - Ax(t)| = 0, \tag{9}$$

and

$$|x'(t) - x'(s)| \leqslant L|t-s| \tag{10}$$

for every $t, s \in \mathbb{R}_+$ and some $L > 0$, there is a C^1 vector function α with the Lipschitz first derivative and such that

$$\alpha(t+1) = A\alpha(t), \tag{11}$$

$$\lim_{t\to+\infty} |x(t) - \alpha(t)| = 0. \tag{12}$$