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Piecewise linear lower and upper bounds for the standard normal first order loss function

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ABSTRACT

The first order loss function and its complementary function are extensively used in practical settings. When the random variable of interest is normally distributed, the first order loss function can be easily expressed in terms of the standard normal cumulative distribution and probability density function. However, the standard normal cumulative distribution does not admit a closed form solution and cannot be easily linearised. Several works in the literature discuss approximations for either the standard normal cumulative distribution or the first order loss function and their inverse. However, a comprehensive study on piecewise linear upper and lower bounds for the first order loss function is still missing. In this work, we initially summarise a number of distribution independent results for the first order loss function and its complementary function. We then extend this discussion by focusing first on random variables featuring a symmetric distribution, and then on normally distributed random variables. For the latter, we develop effective piecewise linear upper and lower bounds that can be immediately embedded in MILP models. These linearisations rely on constant parameters that are independent of the mean and standard deviation of the normal distribution of interest. We finally discuss how to compute optimal linearisation parameters that minimise the maximum approximation error.

1. Introduction

Consider a random variable ω and a scalar variable x. The first order loss function is defined as

$$\mathcal{L}(x,\omega) = \mathrm{E}[\max(\omega - x, 0)], \tag{1}$$

where E denotes the expected value. The complementary first order loss function is defined as

$$\widehat{\mathcal{L}}(x,\omega) = \mathrm{E}[\max(x - \omega, 0)]. \tag{2}$$

The first order loss function and its complementary function play a key role in several application domains. In inventory control [13] it is often used to express expected inventory holding or shortage costs, as well as service level measures such as the widely adopted "fill rate", also known as β service level [1, p. 94]. In finance the first order loss function may be employed to

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capture risk measures such as the so-called "conditional value at risk" (see e.g. [11]). These examples illustrate possible applications of this function. Of course, the applicability of this function goes beyond inventory theory and finance.

In Section 2, we first summarise a number of distribution independent results for the first order loss function and its complementary function. We then focus on symmetric and normal distributions, for which we discuss ad hoc results in Section 3.

According to one of these results, the first order loss function can be expressed in terms of the cumulative distribution function of the random variable under scrutiny. Depending on the probability distribution adopted, integrating this function may constitute a challenging task. For instance, if the random variable is normally distributed, no closed formulation exists for its cumulative distribution function. Several approximations have been proposed in the literature (see e.g. [3–5,10,12,16,17]), which can be employed to approximate the first order loss function. However, these approximations are generally nonlinear and cannot be easily embedded in mixed integer linear programming (MILP) models.

In Sections 4 and 5, we introduce piecewise linear lower and upper bounds for the first order loss function and its complementary function for the case of normally distributed random variables. These bounds are based on standard bounding techniques from stochastic programming, i.e. Jensen's lower bound and Edmundson–Madansky upper bound [9, pp. 167–168]. The bounds can be readily used in MILP models and do not require instance dependent tabulations. Our linearisation strategy is based on standard optimal linearisation coefficients computed in such a way as to minimise the maximum approximation error, i.e. according to a minimax approach – see [7,8,15] for a similar approach. Optimal coefficients for approximations comprising from two to eleven segments will be presented in Tables 1 and 2; these can be reused to approximate the loss function associated with any normally distributed random variable.

2. The first order loss function and its complementary function

Consider a continuous random variable ω with support over $\mathbb{R}$, probability density function $g_\omega(x) : \mathbb{R} \to (0,1)$ and cumulative distribution function $G_\omega(x) : \mathbb{R} \to (0,1)$. The first order loss function can be rewritten as

$$\mathcal{L}(x,\omega) = \int_{-\infty}^{\infty} \max(t-x,0) g_\omega(t)\,dt = \int_{x}^{\infty} (t-x) g_\omega(t)\,dt. \tag{3}$$

The complementary first order loss function can be rewritten as

$$\widehat{\mathcal{L}}(x,\omega) = \int_{-\infty}^{\infty} \max(x-t,0) g_\omega(t)\,dt = \int_{-\infty}^{x} (x-t) g_\omega(t)\,dt. \tag{4}$$

We introduce the following two well-known lemmas.

Lemma 1 [14, p. 338, C.3]. *The first order loss function $\mathcal{L}(x,\omega)$ can also be expressed as*

$$\mathcal{L}(x,\omega) = \int_{x}^{\infty} (1 - G_\omega(t))\,dt. \tag{5}$$

Lemma 2 [14, p. 338, C.4]. *The complementary first order loss function $\widehat{\mathcal{L}}(x,\omega)$ can also be expressed as*

$$\widehat{\mathcal{L}}(x,\omega) = \int_{-\infty}^{x} G_\omega(t)\,dt. \tag{6}$$

There is a close relationship between the first order loss function and the complementary first order loss function.

Lemma 3 [14, p. 338, C.5]. *The first order loss function $\mathcal{L}(x,\omega)$ can also be expressed as*

$$\mathcal{L}(x,\omega) = \widehat{\mathcal{L}}(x,\omega) - (x - \tilde{\omega}), \tag{7}$$

where $\tilde{\omega} = \mathrm{E}[\omega]$.

Because of the relation discussed in Lemma 3, in what follows without loss of generality most of the results will be presented for the complementary first order loss function.

Another known result for the first order loss function and its complementary function is their convexity, which we present next.

Lemma 4. *$\mathcal{L}(x,\omega)$ and $\widehat{\mathcal{L}}(x,\omega)$ are convex in x.*

Proof. It is sufficient to show that $\frac{d^2}{dx^2}\widehat{\mathcal{L}}(x,\omega)$ is positive; furthermore, the proof for $\mathcal{L}(x,\omega)$ follows from Lemma 3 and from the fact that $-x$ is convex. □

For a random variable ω with symmetric probability density function, we introduce the following results.