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The dynamic of plankton–nutrient interaction with delay

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ABSTRACT

In this paper we consider the plankton–nutrient interaction model with the help of delay differential equations. Firstly the elementary dynamical properties of the plankton–nutrient system in the absence of time delay is discussed. Then we establish the existence of local Hopf-bifurcation as the time delay crosses a threshold value. Explicit results are derived for stability and direction of the bifurcating periodic solution by using normal form theory and center manifold arguments. Finally, outcomes of the system are validated through numerical simulations and complex dynamic of the system is explored with the existence of chaotic attractors.

1. Introduction

A feature of plankton populations is the occurrence of rapid population explosions and almost equally rapid declines, separated by periods of almost stationary high or low population levels generally known as “bloom” or “algal bloom”. Generally, highly nutrient and favorable conditions play a key role in rapid or massive growth of algae and low nutrient concentration as well as unfavorable conditions inevitably limits their growth. Although the sudden appearance and disappearance of blooms is not well understood but many researchers have studied the nutrient–plankton interaction to understand the importance of nutrient concentration on the growth of plankton [1–8]. The persistence and co-existence of nutrient–plankton interaction have also been discussed by Ruan [9,10]. Recently Wang et al. [11] proposed a nutrient–plankton model system for a water ecosystem and has studied its global dynamics under different levels of nutrient concentration. The understanding of the dynamic of plankton–nutrient system becomes complex when additional effects of toxicity (caused due to the release of toxin substances by some phytoplankton species known as harmful phytoplankton) are considered. The role of toxin and nutrient on the plankton system have been discussed in [12–18]. Sarkar et al. (see [19,20]) studied the interaction of toxin producing phytoplankton-zooplankton system and concluded that harmful phytoplankton may be used as bio-control agent in the termination of harmful planktonic blooms. It is well known that time delay in biological systems is a reality and it can have complex impact on the dynamic of the system namely loss of stability, induced oscillations and periodic solutions [21–23]. The interaction of plankton–nutrient model with delay due to gestation and nutrient recycling has also been studied by Ruan [24] and Das [25]. Chattopadhyay et al. [26] proposed and analysed a mathematical model of toxic phytoplankton (Noctilucca Scintillans belonging to the group Dinoflagellates of the division Dinophyta)-zooplankton (Paracalanus belonging to the group Copepoda) interaction and assumed that the liberation of toxic substances by the phytoplankton species is not an instantaneous process but is mediated by some time lag required for maturity of species. Extending the work of [26], Bandyopadhyay et al. [27] and Rehim et al. [28] have studied the global stability of the toxin producing phytoplankton-zooplankton system. Sufficient efforts have already been made to understand the interaction of

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phytoplankton-zooplankton system with delay in toxin liberation, but the study of nutrient–plankton interaction with delay in toxin liberation by the phytoplankton species is not done so far. In this paper, an open system with three interacting components consisting of phytoplankton (x), zooplankton (y) and dissolved nutrient (N) is considered. Here, it is assumed that the functional form of biomass conversion by the herbivore is of Holling type-II and the predator is obligate that is they does not take nutrient directly. The toxic substance term which causes extra mortality in zooplankton is expressed by Holling type-II functional response [17,29]. It is also taken into account that the liberation of toxic substances by the phytoplankton species follows discrete time variation.

The main aim of the present study is to see the effect of this discrete time delay on the nutrient–plankton interaction and the organisation of our paper is as follows: In the next section the model equations for the given system are developed. The stability of the co-existence equilibrium E_* in the absence of delay is discussed in the Section 3. After that in Section 4, the delayed plankton model is considered and taking delay as bifurcation parameter the dynamical behaviour of the system around coexisting equilibrium is discussed. The direction and stability of the bifurcating solution using a technique based upon normal form theory and center manifold theorem is determined in Section 5. Some support to our analytical findings through numerical simulations are given in Section 6. Finally, we mention basic outcomes of our mathematical findings and their ecological significance in the concluding section.

2. The mathematical model

Let $N(t)$ denotes the concentration of nutrient at time 't'. Let $x(t)$ and $y(t)$ be the concentration of phytoplankton and zooplankton population respectively at time 't'. Let N_0 is the constant input of nutrient concentration and a is their absorption rate. Let b and α_1 be the nutrient uptake rate for the phytoplankton population and conversion rate of nutrient for the growth of phytoplankton population, respectively ($b \leqslant \alpha_1$). Let β be the maximal zooplankton ingestion rate and β_1 ($\beta_1 \leqslant \beta$) be the maximal zooplankton conversion rate. Let b_1 be the mortality rate of the phytoplankton population, α_2 be the mortality rate of the zooplankton population and let k_1 be the nutrient recycle rate after the death of phytoplankton population. It is also assumed zooplankton population decay at the rate of ρ due to toxin liberation by phytoplankton species. The grazing phenomenon is described by the Holling type-II functional form with γ as the half saturation constant. Let τ is the time delay which is incorporated with the assumption that the liberation of toxin is not instantaneous rather it is mediated by some time lag. The biological significance of this time lag lies in the fact that this time may be considered as the time required for the maturity of toxic-phytoplankton to reduce the grazing impact of zooplankton.

With these assumptions our model system is

$$\begin{cases} \frac{dN}{dt} = N_0 - aN - bNx + k_1 b_1 x \\ \frac{dx}{dt} = \alpha_1 Nx - b_1 x - \frac{\beta xy}{(\gamma + x)} \\ \frac{dy}{dt} = \frac{\beta_1 xy}{(\gamma + x)} - \alpha_2 y - \rho \frac{x(t-\tau)y}{\gamma + x(t-\tau)} \end{cases} \tag{2.1}$$

The initial conditions of the system (2.1) has the form $N(\theta) = \phi_1(\theta)$, $x(\theta) = \phi_2(\theta)$, $y(\theta) = \phi_3(\theta)$, $\phi_1(\theta) \geqslant 0$, $\phi_2(\theta) \geqslant 0$, $\phi_3(\theta) \geqslant 0$, $\theta \in [-\tau, 0]$, $\phi_1(0) \geqslant 0$, $\phi_2(0) \geqslant 0$, $\phi_3(0) \geqslant 0$, where $\phi_1(\theta), \phi_2(\theta), \phi_3(\theta) \in C([-\tau, 0], R^3_+)$, the banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_+ where $R^3_+ = \{(x_1, x_2, x_3) : x_i \geqslant 0, \; i = 1, 2, 3\}$.

3. Analysis of the model without delay

The given model system (2.1) has following equilibria:

(i) The boundary equilibrium $E_1 = (\frac{N_0}{a}, 0, 0)$,
(ii) A planar equilibrium $E_2 = (\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b - \alpha_1 k_1)}, 0)$ exist if $N_0 > \frac{ab_1}{\alpha_1}, \alpha_1 < \frac{b}{k_1}$ and
(iii) A positive interior equilibrium $E_* = (N_*, x_*, y_*)$, where $N_* = \frac{N_0 + k_1 b_1 x_*}{a + bx_*}$, $x_* = \frac{\alpha_2\gamma}{\beta_1 - \rho - \alpha_2}$ and $y_* = \frac{(\alpha_1 N_* - b_1)(\gamma + x_*)}{\beta}$ exist if $\beta_1 > \rho + \alpha_2$ and $N_* > \frac{b_1}{\alpha_1}$.

Proposition 3.1. The plankton free equilibrium $E_1 = (\frac{N_0}{a}, 0, 0)$ always exists and is stable as long as the constant input rate of nutrient is less than certain threshold value, i.e., $N_0 < \frac{ab_1}{\alpha_1}$. Moreover zooplankton free equilibrium, i.e., $E_2 = (\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b - \alpha_1 k_1)}, 0)$ exists and is unstable if the growth rate of phytoplankton biomass α_1 satisfies the inequality, $\alpha_1 < min(\frac{ab_1}{N_0}, \frac{b}{k_1})$.

Definition. The equilibrium E_* is called asymptotically stable if there exist a $K > 0$ such that $\sup_{-\tau \leqslant \theta \leqslant 0}[|\phi_1(\theta) - N_*| + |\phi_2(\theta) - x_*| + |\phi_3(\theta) - y_*|] < \delta$ which implies that $\lim_{t \to \infty} (N(t), x(t), y(t)) = (N_*, x_*, y_*)$, where $(N(t), x(t), y(t))$ is the solution of the system (2.1) with given initial conditions.

The characteristic equation of the system at E_* has the following form

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D + E\lambda)e^{-\lambda\tau} = 0 \tag{3.1}$$