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A subtly analysis of Wilker inequality

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ABSTRACT

The aim of this work is to improve Wilker inequalities near the origin and $\pi/2$.

1. Introduction and motivation

In 1989, Wilker [8] presented the following inequality for $x \in (0, \pi/2)$

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2.$$

This inequality is of great practical importance and it was extended in different forms in the recent past. We refer to [1–14] and all references therein. As Wilker [8] asked about the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x, \quad x \in (0, \pi/2),$$

Sumner et al. [7] proved the following sharp inequality for $x \in (0, \pi/2)$

$$2 + \frac{16}{\pi^4} x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x. \tag{1}$$

Constants $8/45$ and $16/\pi^4$ are somehow motivated, since they are the limits at 0, respective $\pi/2$ of the function

$$x \mapsto \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}.$$

Recently, Chen and Cheung [1] proved that this function decreases monotonically on $(0, \pi/2)$ from $8/45$ to $16/\pi^4$. We have

$$\lim_{x \to 0_+} \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} = \frac{8}{45} \tag{2}$$

and

$$\lim_{x \to (\pi/2)_-} \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x} = \frac{16}{\pi^4}. \tag{3}$$

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It is true that inequalities (1) and some of recent improvements are nice through their symmetric form, but let us remember the practical importance of an inequality which is to provide some bounds for a given expression. In case of (1) observe that near $\pi/2$, the right-hand side inequality becomes weak, since

$$\lim_{x\to(\pi/2)_-}\left(\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-\left(2+\frac{8}{45}x^3\tan x\right)\right)=-\infty.$$

As a consequence, if we are interested in finding good approximations of expression $\left(\sin^2 x\right)/x^2+(\tan x)/x$ in terms of Wilker inequality, then near zero, the constant $8/45$ should be used, while near $\pi/2$, the best choice is $16/\pi^4$.

In other words, 2,3 show us that good approximations near zero are obtained of the form

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\approx 2+\left(\frac{8}{45}+\lambda(x)\right)x^3\tan x,$$

with $\lambda(x)\to 0$, as $x\to 0$, while good approximations near $\pi/2$ are obtained of the form

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\approx 2+\left(\frac{16}{\pi^4}+\mu(x)\right)x^3\tan x,$$

with $\mu(x)\to 0$, as $x\to\pi/2, x<\pi/2$.

2. Results

In the light of the discussion from the previous section, we propose the following new results.

Numerical computations show that both lower and upper bounds in (1) have values pretty close to the value of the middle term, when x is positive and close to zero. But (1) can be improved in the following form.

Theorem 1. *For every $x\in(0,1)$, we have*

$$2+\left(\frac{8}{45}-a(x)\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\left(\frac{8}{45}-b(x)\right)x^3\tan x, \tag{4}$$

where

$$a(x)=\frac{8}{945}x^2,\quad b(x)=\frac{8}{945}x^2-\frac{16}{14175}x^4.$$

We expect the inequalities in (4) to give better results than (1), if we have in mind (2). This is true, since we have for positive values of x near the origin:

$$\frac{8}{45}-b(x)=\frac{8}{45}-\frac{8}{14175}x^2\left(15-2x^2\right)<\frac{8}{45},\quad x\in\left(0,\sqrt{\frac{15}{2}}\right)$$

and

$$\left(\frac{8}{45}-a(x)\right)-\frac{16}{\pi^4}=\frac{8}{945}\left(21-\frac{1890}{\pi^4}-x^2\right)>0,\quad x\in\left(0,\sqrt{21-\frac{1890}{\pi^4}}\right).$$

Theorem 2. *For every $x\in\left(\frac{\pi}{2}-\frac{1}{3},\frac{\pi}{2}\right)$ in the left-hand side, and for every $x\in\left(\frac{\pi}{2}-\frac{1}{2},\frac{\pi}{2}\right)$ in the right-hand side, the following inequalities hold true:*

$$2+\left(\frac{16}{\pi^4}+c(x)\right)x^3\tan x<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}<2+\left(\frac{16}{\pi^4}+d(x)\right)x^3\tan x \tag{5}$$

where

$$c(x)=\left(\frac{160}{\pi^5}-\frac{16}{\pi^3}\right)\left(\frac{\pi}{2}-x\right),\quad d(x)=c(x)+\left(\frac{960}{\pi^6}-\frac{96}{\pi^4}\right)\left(\frac{\pi}{2}-x\right)^2.$$

Now we can see that (5) is sharper than (1) for values of $x<\pi/2$ and close to $\pi/2$, since

$$\left(\frac{16}{\pi^4}+c(x)\right)-\frac{16}{\pi^4}=\frac{16}{\pi^5}\left(10-\pi^2\right)\left(\frac{\pi}{2}-x\right)>0,\quad x<\frac{\pi}{2}.$$