Applied Mathematics and Computation 231 (2014) 560–565

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some invariant solutions of field equations with axial symmetry for empty space containing an electrostatic field

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ARTICLE INFO

Keywords:
General relativity
Electrostatic field
Exact solutions
Symmetry analysis

ABSTRACT

The system of partial differential equations corresponding to a line element with axial symmetry for empty space containing an electrostatic field has been examined. The symmetries of field equations are obtained to derive some ansätz leading to the reduction of variables and some exact solutions are furnished by considering the optimal system.

1. Introduction

Symmetries in general relativity have been the subject of much study in recent years, partly because of the considerable simplification of Einstein's equations resulting from the assumption of one or more symmetries, partly because of interest in the geometric significance of the symmetries, which are described by vector fields of certain geometrical objects on the manifold, and partly because of the possible physical significance of the existence of these symmetries. The literature abounds with many different techniques that have been invoked in an effort to obtain new exact solutions of Einstein field equations [1].

In the general theory of relativity the field equations for regions containing electromagnetic fields but no matter are [2] as follows:

$$\begin{aligned} G_{\mu\nu} &= -8\pi E_{\mu\nu} \\ E^{\nu}_{\mu} &= -F^{\nu\alpha}F_{\mu\alpha} + \tfrac{1}{4}g^{\nu}_{\mu}F^{\alpha\beta}F_{\alpha\beta}, \end{aligned} \tag{1.1}$$

where g^{ν}_{μ} is metric tensor, $G_{\mu\nu}$ is the contracted Riemann–Christoffel tensor, and $F_{\mu\nu}$ is the electromagnetic field tensor. This last tensor satisfies Maxwell's equations if we write

$$\begin{aligned} F_{\mu\nu} &= \kappa_{\mu,\nu} - \kappa_{\nu,\mu} \\ \mathfrak{I}^{\mu\nu}_{,\nu} &= \mathfrak{I}^{\mu}, \end{aligned} \tag{1.2}$$

where κ_{μ} is the four-potential, and $\mathfrak{I}^{\mu}$ is the charge-and-current density which is equal to zero for the region free of matter.

Weyl [3] has found a class of solutions of the above equations corresponding to certain axially symmetric electrostatic fields. In such fields the potential has only one non-vanishing component κ_4 (denoted hereafter by $\frac{1}{2}\pi^{\frac{-1}{2}}\phi$). Weyl's solution is for the axially symmetric case where there is a functional relation between g_{44} and ϕ of the form

$$g_{44} = A + B\phi + \phi^2, \tag{1.3}$$

where A and B are arbitrary constants.

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.12.120

Majumdar [4] and Papapetrou [5] have considered solutions of Eqs. (1.1) and (1.2), when no spatial symmetry is assumed, and have given the general solution when there is a relationship between g_{44} and ϕ of the form

$$g_{44} = (C + \phi)^2, \tag{1.4}$$

where C is a constant. Majumdar has also proved that (1.3) is the only possible functional relationship between g_{44} and ϕ, whether or not there is spatial symmetry.

With the above-mentioned exception, the only exact electrostatic solutions reported appear to be special cases of Weyl's solution. Among the latter are the following: the well-known solution for a charged mass-point [2], the axially symmetric solution of Curzon [6] for several charged mass-points when the relation (1.4) exists between g_{44} and ϕ, the case of an electric field of uniform direction studied by McVittie [7], the solution of Mukherji [8] for a charged line-mass, and a solution corresponding to a particular uniform electric field given by Papapetrou [5].

In this work, we will derive certain axially symmetric electrostatic solutions of the field equations for empty space. We use the canonical cylinder coordinates introduced by Weyl and obtained complete sets of solutions in these coordinates.

In canonical coordinates, the line element for a field with axial symmetry is

$$ds^2 = -\exp(\lambda)(dx_1^2 + dx_2^2) - \exp(-\rho)x_2^2 dx_3^2 + \exp(\rho)dx_4^2, \tag{1.5}$$

where the origin of coordinates is on the axis of symmetry x_1, x_2 is a radial coordinate, x_3 is an angular coordinate and x_4 is time-like. λ and ρ are functions of x_1 and x_2 only. Eqs. (1.1) and (1.2), with $\Im^\mu = 0$, yield the following set which has previously been given by [6]:

$$\lambda_{11} + \lambda_{22} + \rho_1^2 + \frac{\lambda_2}{x_2} = 2\exp(-\rho)(\phi_1^2 - \phi_2^2), \tag{1.6}$$

$$\lambda_{11} + \lambda_{22} + \rho_2^2 - \frac{\lambda_2}{x_2} - \frac{2\rho_2}{x_2} = -2\exp(-\rho)(\phi_1^2 - \phi_2^2), \tag{1.7}$$

$$\rho_1\rho_2 - \frac{\rho_1}{x_2} - \frac{\lambda_1}{x_2} = 4\exp(-\rho)\phi_1\phi_2, \tag{1.8}$$

$$\rho_{11} + \rho_{22} + \frac{\rho_2}{x_2} = 2\exp(-\rho)(\phi_1^2 + \phi_2^2), \tag{1.9}$$

$$\phi_{11} + \phi_{22} + \frac{\phi_2}{x_2} = (\rho_1\phi_1 + \rho_2\phi_2), \tag{1.10}$$

where the suffixes 1 and 2 after λ, ρ and ϕ means partial differentiation with respect to x_1 and x_2.

So, we have five Eqs. (1.6)–(1.10) for the determination of three unknowns λ, ρ and ϕ, one can easily verify that these all are consistent. We will first solve Eqs. (1.9) and (1.10) for ρ and ϕ. It may be pointed out that Eqs. (1.9) and (1.10) are a set of coupled, second order, nonlinear partial differential equations in ρ and ϕ, hence we will concentrate on these two equations and λ can later be obtained from (1.6)–(1.8), once ρ and ϕ are known. Rewriting Eqs. (1.9) and (1.10)

$$\begin{aligned} &\rho_{11} + \rho_{22} + \tfrac{\rho_2}{x_2} = 2\exp(-\rho)(\phi_1^2 + \phi_2^2), \\ &\phi_{11} + \phi_{22} + \tfrac{\phi_2}{x_2} = (\rho_1\phi_1 + \rho_2\phi_2). \end{aligned} \tag{1.11}$$

Due to nonlinearity of exponential order, it is difficult to obtain exact solutions of the system (1.11) and the exact solutions of the system (1.11) may enable one to better understand the phenomena which it describes. A detailed systematic analysis that leads to an exact analytic solution or numerical solutions for (1.11) has not been performed in literature and is therefore desirable. Thus, the importance of the system (1.11) and the need to have some exact solutions are the main motives behind the present study.

Exact solutions, as opposed to computer generated solutions, are of immense importance in understanding of physical problems. The strong desire of exact and more general solutions to Einstein field equations in technological enhancement and for research purpose made tremendous growth in the field of finding exact solutions. Exact solutions of Einstein field equations, which can be compared with approximate or numerical results, are useful in checking validity of approximation techniques and programs. Therefore, finding exact solutions of Einstein field equations is one of the major task.

In order to avoid the cumbersome process of solving nonlinear equations, different solution techniques are used to find solution of field equations but mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. Lie classical method can be categorized in this class and generally a variety of exact solutions may be furnished in a systematic manner.

Lie symmetry methods are central to the modern approach for studying nonlinear differential equations. They use the notion of symmetry to generate solutions in a systematic manner. This theory enables to derive solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses. Thus, it is very powerful, versatile method and generate exact solutions in a very systematic manner as compared to other methods. Applications of Lie Group