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Comparison of probabilistic algorithms for analyzing the components of an affine algebraic variety

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ABSTRACT

Systems of polynomial equations arise throughout mathematics, engineering, and the sciences. It is therefore a fundamental problem both in mathematics and in application areas to find the solution sets of polynomial systems. The focus of this paper is to compare two fundamentally different approaches to computing and representing the solutions of polynomial systems: numerical homotopy continuation and symbolic computation. Several illustrative examples are considered, using the software packages Bertini and Singular.

1. Introduction

Systems of polynomial equations arise throughout mathematics and its application areas as a means of describing, restricting, encoding, or modeling various aspects of a given problem. The areas where polynomial systems have been utilized is incredibly diverse, touching nearly every branch of mathematics and nearly every branch of science and engineering. One is typically interested in understanding some feature of the common set of zeros of the system with the level of detail required being very much dependent on the context and source of the problem. At the most basic level, one would like to understand if the system is consistent, i.e., whether the set of solutions is empty or non-empty. If the solution set is non-empty, the next level is to decide whether the solution set is finite and, if so, to determine the number of solutions and to represent each solution (or each solution together with its multiplicity). If there are infinitely many solutions to the polynomial system, then the solution set can be decomposed into equidimensional components which can be further decomposed into irreducible components. At this level, the most basic questions range from determining the dimension of the largest component to representing each reduced, irreducible component together with a description of its degree, dimension, and multiplicity.

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The focus of this paper is to compare two fundamentally different approaches to computing and representing the irreducible components of the common set of zeros of a system of polynomial equations with rational coefficients. The first approach, based on the method of homotopy continuation, is numerical in nature and leads to a representation of each irreducible component with a collection of "witness points" and basic discrete data. The witness points for an irreducible component consist of a collection of numerical approximations to generic points lying on the component while the discrete data includes the dimension and degree, along with some information about the multiplicity. The second approach is symbolic in nature and leads to a representation of each irreducible component with an ideal and basic discrete data. The ideal is described by a set of generators whose common zeros correspond to points on the irreducible component while the discrete data also includes the dimension and degree, along with some information about the multiplicity. Beyond the dimension, degree, and multiplicity, there are many other discrete and continuous invariants that can be considered useful as a description of the features of (ensembles of) irreducible components of a system of polynomial equations. While very interesting, these finer invariants go beyond the scope of this paper.

As described above, the aim of this paper is to compare two very different approaches to decomposing a polynomial system and to explore their strengths and weaknesses. In Section 2, we briefly discuss basic numerical algebraic geometry, leading to an explanation of the numerical irreducible decomposition algorithm in Section 3. In Section 4, we give a description of two algorithms that are used to symbolically compute a related algebraic decomposition. More specifically, this is the decomposition of the radical of the ideal I as an intersection of prime ideals which contain I. In Section 5, we compare the two different approaches on a collection of benchmark examples. In particular, we compare running times for homotopy-based numerical decomposition algorithms over $\mathbb{C}$ with running times for symbolic decomposition algorithms over $\mathbb{Q}$ for systems of polynomial equations with rational coefficients. For the numerical runs, we use the software package Bertini [5] on a single processor and also on a multiprocessor system with 64 cores. The symbolic runs are performed using the Singular software package [11] on a single processor.

Our findings make it clear that both symbolic and numerical based decomposition methods have strengths that can and should be combined to create improved and more flexible decomposition algorithms. Strengths of numerical based decomposition methods include parallelizability, parameter homotopies, and the ability to extract meaningful information from systems presented with floating point coefficients. Strengths of symbolic based decomposition methods include exact output, the ability to exploit certain special structure in a polynomial system, and the ability to obtain very fine invariants such as syzygies and cohomology modules. A fully integrated system that can take advantage of the strengths of each approach while avoiding the weaknesses is clearly a goal that should be pursued. However, such a hybrid system and the development of novel numeric-symbolic algorithms is beyond the scope of this paper and are the topic of future work.

2. Numerical algebraic geometry

Fix a system

$$f(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{bmatrix}, \tag{1}$$

of n polynomials with $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$.

The numerical methods of the field of *numerical algebraic geometry* may be used to compute accurate approximations of the complex solution set of $f(z)$. This includes both the set of isolated solutions (points) and a numerical analogue of the irreducible decomposition (positive-dimensional solution sets). Though these numerical methods sacrifice the certainty intrinsic to symbolic methods, they reliably produce useful solutions to problems that may be intractable with symbolic methods.

The principal notion of the field is homotopy continuation, which will produce a superset of all isolated solutions of a given system $f(z) = 0$ with $n = N$. The idea is to construct an easy to solve system $g(z) = 0$ and then glue $g(z)$ to $f(z)$ with a new parameter t, to form a homotopy function of N polynomials $H(z,t)$ in $N + 1$ variables, $(z,t) = (z_1, \ldots, z_N, t) \in \mathbb{C}^{N+1}$, such that

1. $H(z,0) = f(z)$ and $H(z,1) = g(z)$;
2. for each solution z^* of $g(z) = 0$ there is a real analytic path $s : (0,1] \to \mathbb{C}^N$ such that $H(s(t),t) = 0$ for $t \in (0,1]$ and $s(1) = z^*$;
3. for each $t^* \in (0,1]$, the Jacobian of $H(z,t^*)$ is nonsingular; and
4. the finite set S of the limits $\lim_{t\to 0} s(t)$ of all the paths includes all isolated solutions of $f(z) = 0$.

There are many ways to construct the start systems $g(z)$ and to follow the solution paths to find the limits. See [8,24,35] for detailed developments of this theory. Among the various start system options, the Bertini software package provides total degree and multihomogeneous homotopies.

So far, we merely have existence of the path $s(t)$, i.e., we do not know the equations of $s(t)$. Whereas the underlying theory relies heavily on ideas from several complex variables and algebraic geometry, the computations primarily involve numerical linear algebra. Tracking the path is an exercise in numerical computation – primarily predictor–corrector methods – for