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# On block-diagonally preconditioned accelerated parameterized inexact Uzawa method for singular saddle point problems ☆

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ABSTRACT

Recently, [Z.-Z. Bai, Z.-Q.Wang, On parameterized inexact Uzawa methods for generalized saddle point problems, Linear Algebra Appl. 428 (2008) 2900–2932] studied a class of parameterized inexact Uzawa (PIU) methods and proposed a generalized and modified accelerated parameterized inexact Uzawa (APIU) iteration method for solving nonsingular saddle point problems. In this paper, we further generalize this method to obtain the block-diagonally preconditioned accelerated parameterized inexact Uzawa (BDP–APIU) method for solving singular saddle point problems. Theoretical analysis shows that the semi-convergence of this new method can be guaranteed. In addition, the quasi-optimal parameters of the new method are discussed. Numerical example is given to show the feasibility and effectiveness of the new method for solving singular saddle point problems.



## 1. Introduction

We consider an iterative solution of the large, spare and singular saddle point problem

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ has $rank(B) = r < n$ with $m > n$, and $b \in \mathbb{R}^m, q \in \mathbb{R}^n$ are given vectors. We denote $A^T$ and $A^*$ the transpose and the conjugate transpose of $A$, respectively. The range and the null spaces of $A$ are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$. In addition, $I_n$ is the identity matrix of order $n$.

The linear system (1.1) typically results from scientific and engineering applications, e.g., discrete approximations of certain partial differential equations [2,15,18], constrained weighted least-squares estimation [12], interior point methods in constrained optimization [10] and so on. See [9] for more detailed discussion about saddle point problems.

When the coefficient matrix of (1.1) is nonsingular, i.e., $B$ is of full rank, a large amount of work has been developed to solve the linear system, including Uzawa-type methods [8,14,17,20], matrix splitting methods [3–8], preconditioned Krylov subspace iteration methods [13,22], see [2,9] and the reference therein for more details. Though most often the matrix $B$ occurs in the form of full rank, but not always. If $B$ is rank-deficient, the linear system (1.1) is a singular saddle point problem. There are also a large number of numerical methods for solving the singular linear systems. For example, PMINRES [19], PCG [24], Uzawa-type methods [16,26,27] and the HSS-like methods [1,23].

Recently, Bai and Wang [8] studied the parameterized inexact Uzawa (PIU) method and further extended the PIU method to obtain the accelerated parameterized inexact Uzawa (APIU) iteration method for nonsingular saddle point problems:

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$$\begin{cases} x^{(k+1)} = x^{(k)} + \omega P^{-1}(b - Ax^{(k)} - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^T x^{(k)} - q) + \gamma Q^{-1} B^T (x^{(k+1)} - x^{(k)}) \end{cases} \tag{1.2}$$

where $\omega, \tau$ and $\gamma$ with $\omega, \tau \neq 0$ are given relaxation factors, and $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are given nonsingular matrices. In [20], Gao and Kong generalized the PIU method and presented the block diagonally preconditioned PIU (PPIU) method:

$$\begin{cases} x^{(k+1)} = x^{(k)} + \omega P^{-1}(P_1 b - P_1 A x^{(k)} - P_1 B y^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(P_2 B^T x^{(k+1)} - P_2 q) \end{cases} \tag{1.3}$$

where $\omega, \tau \neq 0$ are given relaxation factors, $P, P_1 \in \mathbb{R}^{m\times m}$ and $Q, P_2 \in \mathbb{R}^{n\times n}$ with $P_1$ and $P_2$ being positive definite, are given nonsingular matrices.

In this paper, we further generalize the APIU and PPIU methods to the block-diagonally preconditioned accelerated parameterized inexact Uzawa (BDP–APIU) method for singular saddle point problem (1.1). The corresponding semi-convergence properties under certain conditions are studied. In addition, the quasi-optimal parameters of the new method are discussed. Numerical examples are given to show the feasibility and effectiveness of the new method for solving singular saddle point problems.

The outline of the paper is as follows. After describing the block-diagonally preconditioned accelerated parameterized inexact Uzawa method for singular saddle point problems in Section 2, we analyze the semi-convergence of this method in Section 3. In addition, the quasi-optimal parameters of the BDP–APIU method for solving singular saddle point problems are discussed in this part. In Section 4, numerical examples are given to show the efficiency of the new algorithm. Finally in Section 5, we give some brief concluding remarks.

## 2. Block-diagonally preconditioned accelerated parameterized inexact Uzawa iterative method

The singular saddle point problem (1.1) can be rewritten as the following equivalent form

$$\mathcal{A}X \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix} \equiv c. \tag{2.1}$$

Let

$$\mathcal{P}^{-1} := \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where $P_1 \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{n\times n}$ are positive definite matrices. Left preconditioning the augmented linear system (2.1) by $\mathcal{P}^{-1}$, we get

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} b \\ -q \end{pmatrix},$$

that is,

$$\begin{pmatrix} P_1 A & P_1 B \\ -P_2 B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_1 b \\ -P_2 q \end{pmatrix}. \tag{2.2}$$

For the coefficient matrix of the linear systems (2.2), we make the following matrix splitting

$$\mathcal{A}_1 \equiv \begin{pmatrix} P_1 A & P_1 B \\ -P_2 B^T & 0 \end{pmatrix} = \mathcal{D} - \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{D} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ P_2 B^T & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} P - P_1 A & -P_1 B \\ 0 & Q \end{pmatrix},$$

$P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$, being good approximations to $P_1 A$ and $(P_2 B^T)P^{-1}(P_1 B)$, respectively, are prescribed symmetric positive definitive matrices. Let

$$\Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix}, \quad \Delta = \begin{pmatrix} \epsilon I_m & 0 \\ 0 & \gamma I_n \end{pmatrix}$$

where $\omega, \tau, \epsilon$ and $\gamma$ with $\omega, \tau \neq 0$ are given real numbers.

Based on the above splitting, the new relaxed iteration method for singular saddle point problem (2.1) is given the following form, we call it as the block-diagonally preconditioned accelerated parameterized inexact Uzawa (BDP–APIU) method: