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Block pulse operational matrix method for solving fractional partial differential equation

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ABSTRACT

In this paper, we first introduce two dimensional block pulse functions and the block pulse operational matrices of the fractional order integration. Also the block pulse operational matrices of the fractional order differentiation are obtained. Then we present a computational method based on the above results for solving a class of fractional partial differential equations. Transforming the initial equation into a Sylvester equation. The error analysis of the method is given. The method is computationally attractive and applications are demonstrated by some numerical examples. Moreover, comparing the methodology with the known technique shows that our approach is more efficient and more convenient.

1. Introduction

Fractional differential equations are generalized from integer order ones, which are achieved by replacing integer order derivatives by fractional ones. Compared with integer order differential equations, their advantages are capability of simulating natural physical process and dynamic system more accurate [1–3]. Fractional calculus has become the focus of interest for many researchers in different field of science and engineering. A lot of research has shown the advantageous use of the fractional calculus in the modeling and control of many dynamical systems [4,5]. For example, fractional calculus is applied to fluid-dynamic traffic, continuum and statistical mechanics, frequency dependent damping behavior of many viscoelastic materials, colored noise, economics, control theory, signal processing [6]. One of the main difficulties is how to solve the fractional differential equations, some methods were proposed to solve them. The most commonly used ones are Variational Iteration Method [7], Adomian Decomposition Method [8,9], Generalized Differential Transform Method [10,11], Operational Matrix Method [12], Finite Difference Method [13] and Wavelet Method [14,15].

In this paper, our study focuses on a class of fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}=f(x,t), \tag{1}$$

subject to the initial conditions

$$u(0,t)=u(x,0)=0, \tag{2}$$

where $\partial^\alpha u(x,t)/\partial x^\alpha$ and $\partial^\beta u(x,t)/\partial t^\beta$ are fractional derivative of Caputo sense, $f(x,t)$ is the known continuous function, $u(x,t)$is the unknown function, $0<\alpha,\beta\leqslant 1$.

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There have been several methods for solving the fractional partial differential equation. Podlubny [16] used the Laplace Transform method to solve the fractional partial differential equations with constant coefficients. Odibat [17] applied generalized differential transform method to solve the numerical solution of linear partial differential equations of fractional order.

Block pulse functions (BPFs), a set of orthogonal functions with piecewise constant values, have been studied and applied as a useful tool in the synthesis, analysis and other problems of control in recent years. Because of their clearness in expressions and their simplicity in formulations, these functions may have definite advantages for problems involving integrals and derivatives [18]. In Ref. [19] the author proposed a numerical method based on Haar wavelet for solving the fractional partial differential equation and compared numerical solution with exact solution. In this paper, we give another numerical method based on block pulse operational matrix for solving the fractional partial differential equation.

2. Definitions of fractional derivatives and integrals

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this article [16].

Definition 1. The Riemann–Liouville fractional integral operator J^{α} of order α is given by

$$J^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1}u(T)dT, \quad \alpha>0, \tag{3}$$

$$J^0u(t)=u(t). \tag{4}$$

Its properties as following:

$$(i)\ J^{\alpha}J^{\beta}=J^{\alpha+\beta}, \quad (ii)\ J^{\alpha}J^{\beta}=J^{\beta}J^{\alpha}, \quad (iii)\ J^{\alpha}J^{\beta}u(t)=J^{\beta}J^{\alpha}u(t).$$

Definition 2. The Caputo definition of fractional differential operator is given by

$$D_*^{\alpha}u(t)=\begin{cases}\frac{d^r u(t)}{dt^r} & \alpha=r\in N^+;\\ \frac{1}{\Gamma(r-\alpha)}\int_0^t \frac{u^{(r)}(T)}{(t-T)^{\alpha-r+1}}dT, & 0\leqslant r-1<\alpha<r.\end{cases} \tag{5}$$

The Caputo fractional derivatives of order α is also defined as $D_*^{\alpha}u(t)=J^{r-\alpha}D^r u(t)$, where D^r is the usual integer differential operator of order r. The relation between the Riemann–Liouville operator and Caputo operator is given by the following expressions:

$$D_*^{\alpha}J^{\alpha}u(t)=u(t), \tag{6}$$

$$J^{\alpha}D_*^{\alpha}u(t)=u(t)-\sum_{k=0}^{r-1}u^{(k)}(0^+)\frac{(t-\alpha)^k}{k!}, \quad t>0. \tag{7}$$

3. Two dimensional block pulse functions

Block pulse functions of one dimensional have been widely used for differential and integral equations. More details for block pulse functions of one dimensional are given in Ref. [20]. These conclusions can be extended to the two dimensional block pulse functions.

3.1. Definition and properties

2D-BPFs are defined by

$$\phi_{i_1 i_2}(x,t)=\begin{cases}1, & (i_1-1)h_1\leqslant x<i_1h_1 \quad \text{and} \quad (i_2-1)h_2\leqslant t<i_2h_2,\\ 0, & \text{otherwise},\end{cases} \tag{8}$$

where $i_1 = 1, 2, \ldots, m_1$ and $i_2 = 1, 2, \ldots, m_2$ with positive integer values for m_1, m_2 and $h_1=\frac{T_1}{m_1}$, $h_2=\frac{T_2}{m_2}$, $T_1, T_2 \in N^+$.

They have the following properties:

1. Disjointness

$$\phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t)=\begin{cases}\phi_{i_1,i_2}(x,t), & i_1=j_1 \quad \text{and} \quad i_2=j_2,\\ 0, & \text{otherwise}.\end{cases} \tag{9}$$