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Some more properties of core partial order

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ABSTRACT

In this paper the core partial order introduced by Baksalary and Trenkler has been studied further. New characterizations of the core partial order have been derived. Relationship between the core partial order and some known partial orders has been also investigated.

1. Introduction and preliminaries

The objective of this paper is to investigate the core partial order for its characterizations and its interaction with some known partial orders, for example, the minus order, the sharp order and the star order. Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices and $Y\in\mathbb{C}^{m\times n}$. The symbols Y^* and $\mathcal{C}(Y)$ will respectively denote the conjugate transpose and the column space of Y. The symbol I will denote the identity matrix. For a matrix $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$AXA = A,\quad XAX = X,\quad (AX)^* = AX \text{ and } (XA)^* = XA$$

is called the Moore–Penrose inverse of A and is denoted by $A^\dagger$. A matrix X satisfying $AXA = A$, is called a g-inverse of A and is denoted by A^- and if in addition it also satisfies $XAX = X$, it is called reflexive g-inverse. If X satisfies

$$AXA = A \text{ and } (AX)^* = AX.$$

X is called a least squares g-inverse. Let $A\in\mathbb{C}^{n\times n}$. A matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$AXA = A \text{ and } \mathcal{C}(X^*)\subseteq\mathcal{C}(A^*)$$

is known as a ρ-inverse of A and if X satisfies

$$AXA = A \text{ and } \mathcal{C}(X)\subseteq\mathcal{C}(A),$$

then X is called a χ-inverse of A. A ρ-inverse of A is denoted by A^-_ρ and a χ-inverse by A^-_χ. A g-inverse of A that is a ρ-inverse as well as a χ-inverse is called a $\rho\chi$-inverse and is denoted by $A^-_{\rho\chi}$. Moreover, a ρ-inverse (as also a χ-inverse and therefore a $\rho\chi$-inverse) of A exists if A is of index $\leqslant 1$.

A $\rho\chi$-inverse of a matrix A (whenever it exists) is the unique commuting reflexive g-inverse called its group inverse and is denoted by $A^\#$.

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Two types of unique generalized inverses, namely $A^{-}_{\rho^*\chi} = A^{-}_{\rho\chi}AA^{\dagger}$ and $A^{-}_{\rho\chi^*} = A^{\dagger}AA^{-}_{\rho\chi}$ for a square matrix A of index $\leqslant 1$ were defined by Rao and Mitra Page 97, [11]. The same have been rediscovered by Baksalary and Trenkler in [3] and the g-inverse $A^{-}_{\rho^*\chi} = A^{-}_{\rho\chi}AA^{\dagger}$ has been named the core inverse of A. Clearly such a g-inverse of a matrix exists only when the matrix is of index $\leqslant 1$ and such matrices are known as core matrices.

Definition 1.1 [3]. The core inverse of a square matrix A is a matrix G such that $AG = P_A$ and $\mathcal{C}(G) \subseteq \mathcal{C}(A)$, where P_A is the orthogonal projector onto $\mathcal{C}(A)$.

Since $P_A = AA^{\dagger}$, it follows that the core inverse of a square matrix A is a matrix G such that $AG = AA^{\dagger}$ and $\mathcal{C}(G) \subseteq \mathcal{C}(A)$. The core inverse G of a square matrix A exists if and only if A is of index $\leqslant 1$. It is a reflexive least squares inverse. Moreover, if A is non-singular (A is of index 0), then its core inverse is the usual inverse.

The other g-inverse $A^{-}_{\rho\chi^*} = A^{\dagger}AA^{-}_{\rho\chi}$, (which also exists when the matrix is of index $\leqslant 1$), has properties and theory similar to that of core inverse as noted in [3].

For the sake of convenience, we record the following canonical form [4] of A and its core inverse G given in [3]. This kind of representation of A and G has been found quite useful in studying the core partial order. Let $A \in \mathbb{C}^{n\times n}$ be a matrix of rank r. Then A can be represented in the form

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \ldots \sigma_t I_{r_t})$ is a diagonal matrix, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > \ldots > \sigma_t, r_1 + r_2 + \ldots + r_t = r$ and $K \in \mathbb{C}^{r\times r}, L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$. The core inverse of A(whenever it exits) is denoted by A^{Θ} and is given as

$$A^{\Theta} = U\begin{pmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*. \tag{2}$$

The Moore–Penrose inverse and the group inverse (whenever it exists) of a matrix A having representation (1) are respectively equal to

$$A^{\dagger} = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix}U^* \text{ and} \tag{3}$$

$$A^{\#} = U\begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{pmatrix}U^* \tag{4}$$

as shown in [3].

The following theorem includes some useful properties of the core inverse that will be used in this paper.

Theorem 1.2. *Let A be square matrix of index* $\leqslant 1$. *Then the following hold:*

(i) $AA^{-}A^{\Theta} = A^{\Theta}$, *for each g-inverse* A^{-} *of A.*
(ii) *Whenever core inverse of A exists, it is unique* [3].

Proof.

(i) is easy.
(ii) As shown in Theorem 1(viii) of [3], we have $A^{\Theta}A = A^{\#}A$. So, if G_1 and G_2 are two core inverses of A, then $G_1A = A^{\#}A, AG_1 = AA^{\dagger}$ and $G_2A = A^{\#}A, AG_2 = AA^{\dagger}$. So, $G_1 = G_1AG_1 = G_1AA^{\dagger} = G_1AG_2 = A^{\#}AG_2 = G_2AG_2 = G_2$. □

A complex square matrix is range Hermitian if and only if $\mathcal{C}(A) = \mathcal{C}(A^*)$ if and only if $A^{\Theta} = A^{\dagger} = A^{\#}$ [3]. If A has representation (1), then A is range Hermitian if and only if $L = 0$ as shown in [3].

We now give definitions of some matrix orders frequently used in this paper. We note that some of these matrix orders have been defined over arbitrary field, for instance in [10]. However, for our needs we record these definitions for complex matrices only.

Let A and B be Hermitian matrices of the same order. Then A is below B under the Löwner order (written as $A \leqslant^L B$) if $B - A$ is a non-negative definite matrix. Let A and B be complex rectangular matrices of the same order. Then A is below B under the star order [9](written as $A \overset{*}{\leqslant} B$) if

$$AA^* = BA^* \text{ and } A^*A = A^*B.$$