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# Existence of nondecreasing solutions of a quadratic integral equation of Volterra type

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ABSTRACT

Nondecreasing solutions of the quadratic integral equations is an interesting subject in the area of mathematics, sciences and engineering. Using the theory of measures of noncompactness and applying a new method, we prove the existence of nondecreasing solutions of a quadratic integral equation of Volterra type in $C(I)$.



## 1. Introduction

In this paper, we discuss the following quadratic integral equation of Volterra type

$$x(t) = g(t, x(t))(h(t) + \int_0^t k(t,s) f(s, x(\lambda s))ds), \quad t \in I = [0,1], \tag{1.1}$$

where $g, \ f : I \times \Re \to \Re$ are given functions, $\lambda \in (0,1]$.

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, and in traffic theory, see [1–5]. Especially, the so-called quadratic integral equation of Chandrasekher type can very often be encountered in many applications (cf. [6–8]).

The study of quadratic integral equation has received much attention over the last thirty years or so. For instance, Cahlon and Eskin [9] prove the existence of positive solutions in the space $C[0,1]$ and $C^\alpha[0,1]$ of an integral equation of the Chandrasekhar H-equation with perturbation. Argyros [10] investigates a class of quadratic equations with a nonlinear perturbation. Banaś et al. [11] prove a few existence theorems for some quadratic integral equations. Banaś and Rzepka [12] study the Volterra quadratic integral equation on unbounded interval. Banaś and Sadarangani [13] study the solvability of Volterra–Stieltjes integral equation. In [14–16] the authors prove the existence of nondecreasing solutions of a quadratic integral equation. Dhage [17,18] proves an existence theorem for a certain differential inclusions in Banach algebras. Dhage [19] proves the existence results of some nonlinear functional integral equations. The purpose of this paper is to continue the study of those authors [14–16]. By applying a new method, we do not need any contraction condition to ensure the operator $T$ is contraction. We prove the existence results of quadratic integral equations of Volterra type by the theory of measures of noncompactness and fixed point theorem.

The organization of this work is as follows. In Section 2, we recall some definitions and theorems about the measure of noncompactness and fixed point theorem. In Section 3, we give theorems on the existence of nondecreasing continuous solutions of a quadratic integral equation of Volterra type (1.1). Finally, in Section 4, examples are given to show the applications of our results.

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## 2. Preliminaries

Now, we are going to present definitions and basic facts needed further on.

Assume $E$ is a real Banach space with norm $\|\cdot\|$. If $X$ is a nonempty subset of $E$, we denote by $\overline{X}$ and $ConvX$ the closure and the closed convex of $X$. Let us denote by $\Gamma_E$ the family of nonempty bounded subsets of $E$ and by $\Upsilon_E$ its subfamily consisting of all relatively compact sets.

**Definition 2.1** [20]. A function $\mu : \Gamma_E \to [0, \infty)$ is said to be a measure of noncompactness in the space $E$ if it satisfies the following conditions:

(1) : The family $ker\mu = \{X \in \Gamma_E, \mu(X) = 0\}$ is nonempty and $ker(\mu) \subset \Upsilon_E$.
(2) : $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
(3) : $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
(4) : $\mu(\theta X + (1-\theta)Y) \leqslant \theta\mu(X) + (1-\theta)\mu(Y), \forall \theta \in [0,1]$.
(5) : If $\{X_n\}$ is a sequence of closed sets from $\Gamma_E$ such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$, and $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

**Remark 2.2.** The family $ker\mu$ described above is called the kernel of the measure of noncompactness $\mu$. Further facts concerning measure of noncompactness and their properties may be found in [20,21].

Let us suppose that $M$ is a nonempty subset of a Banach space $E$ and the operator $T : M \to E$ is continuous and transforms bounded sets onto bounded ones. We say that $T$ satisfies the Darbo condition (with constant $k \geqslant 0$) with respect to a measure of noncompactness $\mu$ if for any bounded subset $X$ of $M$ we have

$$\mu(TX) \leqslant k\mu(X).$$

If $T$ satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to $\mu$.

**Theorem 2.3** [22]. *Let $Q$ be a nonempty, bounded, closed and convex subset of the Banach space $E$ and $\mu$ a measure of noncompactness in $E$. Let $T : Q \to Q$ be a contraction with respect to $\mu$. Then $T$ has a fixed point in the set $Q$.*

**Remark 2.4.** Under the assumptions of the above theorem, it can be shown that the set $FixT$ of fixed points of $T$ belonging to $Q$ is a member of $ker\mu$.

For our purpose, let us recall the definition of the measure of noncompactness in the space $C(I)$ which will be used in Section 3. This measure was introduced in the paper [23].

Let $C(I)$ denote the space of all real functions defined and continuous on the interval $I = [0,1]$. The space $C(I)$ is furnished with standard norm $\|x\| = max\{|x(t)| : t \in I\}$.

Fix a nonempty and bounded subset $X$ of $C(I)$. For $\varepsilon > 0$, and $x \in X$ denote by $w(x, \varepsilon)$ the modulus of continuity of $x$ defined by

$$w(x,\varepsilon) = sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leqslant \varepsilon\}.$$

Furthermore, put

$$w(X,\varepsilon) = sup\{w(x,\varepsilon), x \in X\},$$

$$w_0(X) = \lim_{\varepsilon\to 0} w(X,\varepsilon).$$

Next, let us define the following quantities:

$$d(x) = sup\{|x(t) - x(s)| - [x(t) - x(s)] : t, s \in I, s \leqslant t\},$$

$$d(X) = sup\{d(x) : x \in X\}.$$

Observe that $d(X) = 0$ if and only if all functions belonging to $X$ are nondecreasing on $I$.

Finally, let

$$\mu(X) = w_0(X) + d(X).$$

It can be showed [23] that the function $\mu$ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $ker\mu$ consists all nonempty and bounded subsets $X$ of $C(I)$ such that functions from $X$ are equicontinuous and nondecreasing on the interval $I$.