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Applications of multivalent prestarlike functions

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ABSTRACT

In the paper we introduce general classes of multivalent analytic functions defined by the subordination. In particular, we define the class of multivalent prestarlike functions. By using the Ruscheweyh's duality theory we investigate convolution properties related to multivalent prestarlike functions and various inclusion relationships between defined classes of functions. Some applications involving well-known classes of functions defined by linear operators are also considered.

Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \{z \in \mathbb{C} : |z| < r\}$. We denote by $\mathcal{A}_p$ $(p \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$ the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

Let $\mathcal{V} \subset \mathcal{A}_0$. We define the *dual set*of $\mathcal{V}$ by

$$\mathcal{V}^* := \{q \in \mathcal{A}_0 : (q * h)(z) \neq 0 \quad (h \in \mathcal{V},\ z \in \mathcal{U})\}.$$

Moreover, let us denote

$$\mathcal{T}(\beta) := \left\{ \frac{(1+xz)}{(1+yz)^{\beta}} : |x| = |y| = 1 \right\} \quad (\beta \geqslant 0),$$

$$\mathcal{H} := [\mathcal{T}(1)]^*, \quad \mathcal{H}^k := \{q^k : q \in \mathcal{H}\} \quad (k \geqslant 0),$$

$$\mathcal{K}(0, \lambda) := \left\{ q \in \mathcal{A}_0 : \Re\left(\frac{zq'(z)}{q(z)}\right) > -\frac{\lambda}{2} \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{K}(k, \beta) := \{qh : q \in \mathcal{H}^k \text{ and } h \in \mathcal{K}(0, \beta - k)\} \quad (0 \leqslant k \leqslant \beta).$$

It is clear that

$$V_1 \subset V_2 \Rightarrow V_2^* \subset V_1^* \tag{2}$$

and

$$0 \leqslant k \leqslant \beta_1 \leqslant \beta_2 \Rightarrow \mathcal{K}(k, \beta_1) \subset \mathcal{K}(k, \beta_2). \tag{3}$$

Let $\phi, \varphi \in \mathcal{A}_p$ and let $h_\alpha, \alpha < p$, be a function univalent and convex in $\mathcal{U}$ with

$$h_\alpha(0) = 1, \quad \Re[p h_\alpha(z)] > \alpha \quad (z \in \mathcal{U}). \tag{4}$$

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We denote by $\mathcal{W}_p(\phi(z),\varphi(z);h_\alpha(z))$ (or simply $\mathcal{W}_p(\phi,\varphi;h_\alpha)$) the classof functions $f\in\mathcal{A}_p$ such that

$$\frac{(\phi*f)(z)}{(\varphi*f)(z)}\prec h_\alpha(z).$$

Moreover, let us denote

$$\mathcal{W}_p(\varphi;h_\alpha):=\mathcal{W}_p\left(\frac{z\varphi'(z)}{p},\varphi;h_\alpha\right),$$
$$\mathcal{W}_p(h_\alpha):=\mathcal{W}_p\left(\frac{z^p}{1-z};h_\alpha\right),$$
$$\mathcal{S}_p^*(\varphi;\alpha):=\mathcal{W}_p\left(\varphi;\frac{p+(p-2\alpha)z}{1-z}\right).$$

In particular, the classes

$$\mathcal{S}_p^*(\alpha):=\mathcal{S}_p^*\left(\frac{z^p}{1-z};\alpha\right),\quad \mathcal{S}_p^c(\alpha):=\mathcal{S}_p^*\left(\frac{z^p(1+(1-p)z)}{p(1-z)^2};\alpha\right)$$

are the classes of *multivalent starlike* functions of order α and *multivalent convex* functions of order α, respectively. By (4) it is clear that

$$\mathcal{W}_p(h_\alpha)\subset\mathcal{S}_p^*(\alpha),\quad \mathcal{W}_p(\varphi;h_\alpha)\subset\mathcal{S}_p^*(\varphi;\alpha) \tag{5}$$

and

$$f\in\mathcal{W}(\varphi;h_\alpha)\Longleftrightarrow f*\varphi\in\mathcal{W}(h_\alpha). \tag{6}$$

The class

$$\mathcal{R}_p(\gamma):=\begin{cases}\mathcal{S}_p^*\left(\frac{z^p}{(1-z)^{2(p-\gamma)}};\gamma\right) & \text{for}\quad \gamma<p\\ \mathcal{W}_p\left(\frac{z^p}{1-z},z^p;\frac{1}{1-z}\right) & \text{for}\quad \gamma=p\end{cases}$$

will be called the class of *multivalent prestarlike* functions of order γ. The class $\mathcal{R}(\gamma):=\mathcal{R}_1(\gamma)$ is the well-know class of *prestarlike* functions of order γ introduced by Ruscheweyh [18]. A simple calculation shows that $f\in\mathcal{R}_p(\gamma)$ if and only if

$$\begin{cases}f(z)*\frac{z^p}{(1-z)^{2(p-\gamma)}}\in\mathcal{S}_p^*(\gamma) & \text{for}\quad \gamma<p\\ \Re\left(\frac{f(z)}{z^p}\right)>\frac{1}{2} & \text{for}\quad \gamma=p\end{cases}.$$

We say that a function $f\in\mathcal{A}_p$ belongs to the class $\mathcal{C}_p(\phi,\varphi;h_\alpha)$ if there exists $g\in\mathcal{W}_p(\varphi;h_\alpha)$ such that

$$\frac{(\phi*f)(z)}{(\varphi*g)(z)}\prec h_\alpha(z).$$

The condition (6) defines the linear operator $J:\mathcal{W}(\varphi;h_\alpha)\to\mathcal{W}(h_\alpha),\ J(f)=f*\varphi$. If $\varphi^{(k)}(0)\neq 0\quad (k=p,p+1,\ldots)$, then the operator J is one-to-one and the class $\mathcal{C}_p(\phi,\varphi;h_a)$ contains the functions $f\in\mathcal{A}_p$ for which there exists a function $g\in\mathcal{W}_p(h_\alpha)$ such that $\frac{(\phi*f)(z)}{g(z)}\prec h_\alpha(z)$. Then, we can omit the function φ as the parameter of the class and we can denote it by $\mathcal{C}_p(\phi;h_\alpha)$ i.e.

$$\mathcal{C}_p(\phi;h_\alpha):=\mathcal{C}_p\left(\phi,\frac{z^p}{1-z};h_\alpha\right). \tag{7}$$

In particular, the class

$$CC:=\mathcal{C}_1\left(\frac{z}{(1-z)^2};\frac{1+z}{1-z}\right)$$

is the well-known class of *close-to-convex*functions with parameter $\beta=0$. It is clear that

$$\mathcal{W}_p(\phi,z;h_\alpha)=\mathcal{C}_p(\phi,z;h_\alpha)\subset\mathcal{C}_p(\phi,\varphi;h_\alpha)\subset\mathcal{C}_p(\phi;h_\alpha). \tag{8}$$

These general classes reduced to well-known subclasses by judicious choices of the parameters. Also, there have been already several works on classes defined by convolution and subordination, notably the work [20], as well as those in [1,3,8,13,25].

For complex parameters $a_1,\ldots,a_q$ and $b_1,\ldots,b_s$ ($r\leqslant s+1;\ r,s\in\mathbb{N}:=\mathbb{N}_0\setminus\{0\},\ b_j\neq 0,-1,-2,\ldots;\ j=1,\ldots,s$), the *generalized hypergeometric function* ${}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;z)$, is defined by