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# Characterizations of the commutators and the anticommutator involving idempotents ☆

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ABSTRACT

In this paper, we characterize the idempotency, the invertibility or generalized invertibility, the positivity and the range relations of the commutator $\mathbf{C} = PQ - QP$ and the anticommutator $\mathbf{D} = PQ + QP$ involving idempotents. Particular attention is paid to the idempotent operators $P$ and $Q$, and some further characteristics when $P$ and $Q$ are positive are derived as well.



## 1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we shall denote by $A^*, \mathcal{N}(A)$ and $\mathcal{R}(A)$ the adjoint of $A$, the null space and the range of $A$, respectively. $A$ is said to be positive if $(Ax, x) \geqslant 0$ for all $x \in \mathcal{H}$. If $A$ is positive, the positive square root of $A$ is denoted by $A^{\frac{1}{2}}$ (see [8,22]). For two operators $P, Q \in \mathcal{B}(\mathcal{H})$, the commutator and the anticommutator of $P$ and $Q$ are the operators

$$\mathbf{C} = PQ - QP \quad \text{and} \quad \mathbf{D} = PQ + QP, \tag{1}$$

respectively [25]. Commutators and anticommutators arise naturally in many aspects of operator theory, and they play an important role in this theory. It is well known that the set of commutators is dense in the set of all operators [4, p. 124].

An operator $T$ is called generalized invertible, if there is an operator $S$ such that (I) $TST = T$.The operator $S$ is not unique in general. In order to guarantee its uniqueness, further conditions have to be imposed. The most likely convenient additional conditions are

$$\text{(II)} \quad STS = S, \quad \text{(III)} \quad (TS)^* = TS, \quad \text{(IV)} \quad (ST)^* = ST, \quad \text{(V)} \quad TS = ST.$$

Elements $S \in \mathcal{B}(\mathcal{H})$ satisfying (I, II, III, IV) are called Moore–Penrose inverses (for short MP-inverses), denoted by $S = T^+$. It is well known that $T$ has the MP-inverse if and only if $\mathcal{R}(T)$ is closed, and the MP-inverse of $T$ is unique (see [2,5,23,24,26,27]). Similarly, (I, II, V)-inverses are called group inverses, denoted by $S = T^\#$. Moreover, (I, II, III, IV, V)-inverses are called EP elements (i.e., $T^+ = T^\#$). One also considers $(\mathrm{I}_k)$ $T^kST = T^k$ with some $k \in \mathbf{Z}^+$. Clearly, $(\mathrm{I}) = (\mathrm{I}_1)$. And $(\mathrm{I}_k$, II, V)-inverses are called Drazin inverses, denoted by $S = T^D$ (see [2,5]), where $k$ is the Drazin index of $T$. If $T$ is Drazin invertible, then the spectral idempotent $T^\pi$ of $T$ corresponding to $\{0\}$ is given by $T^\pi = I - TT^D$.

Let $P$ and $Q$ be idempotents. It is well known that $P - Q$ is invertible if and only if $I - PQ$ and $P + Q$ are invertible [20]. This result was generalized to the Drazin invertible case and had gotten that $P - Q$ is Drazin invertible if and only if $I - PQ$ is Drazin invertible if and only if $P + Q$ is Drazin invertible in [10,21]. Recently, Koliha et al. [21] proved that $\mathbf{C}$ is Drazin invertible if and only if $\mathbf{D}$ is Drazin invertible if and only if $P - Q$ and $PQ$ are Drazin invertible. For arbitrary idempotents $P$ and $Q$, we have the operator matrices

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$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \tag{2}$$

with space decompositions $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Since $Q^2 = Q$, we obtain that

$$Q_1^2 + Q_2Q_3 = Q_1, \quad Q_1Q_2 + Q_2Q_4 = Q_2, \quad Q_3Q_1 + Q_4Q_3 = Q_3, \quad Q_3Q_2 + Q_4^2 = Q_4. \tag{3}$$

In particular, if $P$ and $Q$ are orthogonal projections, then $Q_1$ and $Q_4$ are positive and $Q_3 = Q_2^*$. We use the usual notation $\overline{P} = I - P$ and $\overline{Q} = I - Q$.

In this paper, we will investigate the properties of the commutator $\mathbf{C} = PQ - QP$ and the anticommutator $\mathbf{D} = PQ + QP$, where $P$ and $Q$ are idempotents or orthogonal projections. Various relations among the idempotency, the invertibility or generalized invertibility, the positivity and the range relations are considered. Particular attention is paid to the idempotent operators $P$ and $Q$ and some further characteristics when $P$ and $Q$ are positive are derived as well.

## 2. Some lemmas

The following result is given in [6,7] for matrices. Using the definition of Drazin inverse, the result is extended to operators in $\mathcal{B}(\mathcal{H})$.

**Lemma 2.1** [7, Theorem 2.1]. *Let $M \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ have the operator matrix form*

$$M = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{4}$$

*Then $M$ is Drazin invertible if and only if $AB$ (or $BA$) is Drazin invertible. In this case,*

$$M^{\mathrm{D}} = \begin{pmatrix} 0 & (AB)^{\mathrm{D}}A \\ B(AB)^{\mathrm{D}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A(BA)^{\mathrm{D}} \\ (BA)^{\mathrm{D}}B & 0 \end{pmatrix}.$$

Remark that $M$ is MP-invertible if and only if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. In this case,

$$M^{+} = \begin{pmatrix} 0 & B^{+} \\ A^{+} & 0 \end{pmatrix}.$$

**Lemma 2.2** (*see [9, Theorem 2.7]*). *For $A \in \mathcal{B}(\mathcal{H}), A(I-A)$ is Drazin invertible if and only if $A$ and $I - A$ are Drazin invertible.*

For two orthogonal projections $P$ and $Q$, denote $\mathcal{H}_1 = \mathcal{R}(P) \cap \mathcal{R}(Q), \mathcal{H}_2 = \mathcal{R}(P) \cap \mathcal{N}(Q), \mathcal{H}_3 = \mathcal{N}(P) \cap \mathcal{R}(Q), \mathcal{H}_4 = \mathcal{N}(P) \cap \mathcal{N}(Q)$ and let $\mathcal{H}_5 = \mathcal{R}(P) \cap (\mathcal{H} \ominus (\oplus_{i=1}^4 \mathcal{H}_i))$ and $\mathcal{H}_6 = \mathcal{H} \ominus (\oplus_{i=1}^5 \mathcal{H}_i)$. Then $\mathcal{H}_i \perp \mathcal{H}_j, j \neq i$ and $1 \leqslant i, j \leqslant 6$. We have the following lemma which is useful later.

**Lemma 2.3** (*see [14, Lemma 1]*). *Let $P$ and $Q$ be orthogonal projections. Then $P\mathcal{H}_i \subseteq \mathcal{H}_i$ and $Q\mathcal{H}_i \subseteq \mathcal{H}_i, 1 \leqslant i \leqslant 4$, and $P$ and $Q$ have the following operator matrices*

$$P = I \oplus I \oplus 0 \oplus 0 \oplus I \oplus 0, \quad Q = I \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I-Q_0)^{\frac{1}{2}} & D^*(I-Q_0)D \end{pmatrix} \tag{5}$$

*with respect to the space decomposition $\mathcal{H} = \sum_{i=1}^6 \mathcal{H}_i$, respectively, where $Q_0$ is a positive contraction on $\mathcal{H}_5$, 0 and 1 are not eigenvalues of $Q_0$ and $D$ is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$.*

**Lemma 2.4** (*see [15, Theorem 2.2], [12,17]*). *Let $A, B, C \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{\frac{1}{2}})$; $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(AA^*)$; If $C \geqslant 0$, then $\overline{\mathcal{R}(C^{\frac{1}{2}})} = \overline{\mathcal{R}(C)}$ and $\mathcal{R}(C) \subseteq \mathcal{R}(C^{\frac{1}{2}})$; $\mathcal{R}(C)$ is closed if and only if $\mathcal{R}(C) = \mathcal{R}(C^{\frac{1}{2}})$; $\mathcal{R}(C) = \mathcal{H}$ if and only if $C$ is invertible.*

It is relevant to note here that if $P$ or $Q \in \mathcal{B}(\mathcal{H})$ has bounded generalized inverse, then $\mathcal{R}(P)$ or $\mathcal{R}(Q)$ is closed. In the case that the dimension of $\mathcal{H}$ is finite, the MP-inverses and the Drazin inverses of $PQ$ always exist. But it is not true for the case that the dimension of $\mathcal{H}$ is infinite. For example, define positive diagonal operator $Q_0' \in \mathcal{B}(\mathcal{H})$ by $Q_0' = \frac{1}{2} \oplus \frac{1}{3} \oplus \frac{1}{4} \oplus \cdots \oplus \frac{1}{n} \oplus \cdots$. Then $\mathcal{R}(Q_0')$ is not closed since 0 is the accumulation point of $\sigma(Q_0')$ [8]. Put

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_0' & Q_0'^{\frac{1}{2}}(I-Q_0')^{\frac{1}{2}} \\ Q_0'^{\frac{1}{2}}(I-Q_0')^{\frac{1}{2}} & I - Q_0' \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$. Then $P, Q$ are orthogonal projections and