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Continuous dependence for impulsive functional dynamic equations involving variable time scales

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ABSTRACT

Using a known correspondence between the solutions of impulsive measure functional differential equations and the solutions of impulsive functional dynamic equations on time scales, we prove that the limit of solutions of impulsive functional dynamic equations over a convergent sequence of time scales converges to a solution of an impulsive functional dynamic equation over the limiting time scale.

1. Introduction

The fact that solutions of dynamic equations on time scales depend continuously on time scales is a problem that has been investigated by several researchers. See [1,5,10], for instance. In these papers, the authors prove that the sequence of solutions of the problem

$$\begin{cases} x^{\Delta}(t) = f(x,t), & t \in \mathbb{T}_n, \\ x(t_0) = x_0, & t_0 \in \mathbb{T}_n \end{cases} \tag{1.1}$$

converges uniformly to the solution of the problem

$$\begin{cases} x^{\Delta}(t) = f(x,t), & t \in \mathbb{T}, \\ x(t_0) = x_0, & t_0 \in \mathbb{T}, \end{cases} \tag{1.2}$$

whenever $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$, where $d(\mathbb{T}_n, \mathbb{T})$ denotes the Hausdorff metric or the induced metric from the Fell topology.

To obtain such results, the following conditions on the function f are usually assumed:

- There exists a constant $M > 0$ such that $\|f(x,t)\| \leqslant M$ for every x in a certain subset of the phase space and every $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$.
- There exists a constant $L > 0$ such that $\|f(x,t) - f(y,t)\| \leqslant L\|x - y\|$ for every x and y in a certain subset of the phase space and every $t \in [t_0, t_0 + \eta]_{\mathbb{T}}$.

Here, our goal is to investigate the behavior of solutions of the same initial value problems over different time scales for impulsive functional dynamic equations; that is, we prove that, under certain conditions, the sequence of solutions of the system

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$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), & t \in [t_0, t_0+\eta]_{\mathbb{T}_n}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}_n} \end{cases} \tag{1.3}$$

converges uniformly to the solution of the problem

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), & t \in [t_0, t_0+\eta]_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{1.4}$$

whenever $d(\mathbb{T}_n, \mathbb{T}) \to 0$ as $n \to \infty$. Here, $d(\mathbb{T}_n, \mathbb{T})$ denotes the Hausdorff metric. Our results apply to the Fell topology as well.

We also consider the following conditions on the function f:

- There exists a constant $M > 0$ such that

$$\|f(x_t, t)\| \leqslant M$$

for all $t \in [t_0, t_0+\eta]_{\mathbb{T}}$ and all x in a certain subset of the phase space.

- There exists a constant $L > 0$ such that

$$\left\| \int_{u_1}^{u_2} (f(x_t, t) - f(y_t, t))\mathrm{d}g(t) \right\| \leqslant L \int_{u_1}^{u_2} \|x_t - y_t\|_\infty \,\mathrm{d}g(t)$$

for all $u_1, u_2 \in [t_0, t_0+\eta]_{\mathbb{T}}$ and all x, y in a certain subset of the phase space.

Here, we consider the integral in the sense of Henstock–Kurzweil which is known to integrate highly oscillating functions (see [9], for instance). Thus, the second condition on the indefinite integral of f allows the function f to behave "badly", e.g., f may have many discontinuities or be of unbounded variation, and yet good results can be obtained, as long as its indefinite behaves well enough. Alternatively, one could consider the Riemann or Lebesgue integral.

In order to obtain the continuous dependence result for impulsive functional dynamic equations on time scales involving variable time scales with these conditions, we use a known correspondence between the solutions of impulsive functional dynamic equations on time scales and the solutions of impulsive measure functional differential equations. We also use a correspondence between these solutions and the solutions of measure functional differential equations. For details about these correspondences, see [7].

Further, in order to ensure the convergence of solutions, we suppose some convergence over a operator sequence defined in Section 3. This hypothesis cannot be suppressed as shown by Examples 5.1 and 5.2 in Section 5.

2. Impulsive measure functional differential equations

Let $r, \eta > 0$ be given numbers and $t_0 \in \mathbb{R}$. The theory of functional differential equations (see e.g., [8]) deals with problems as

$$\dot{x} = f(x_t, t), \quad t \in [t_0, t_0+\eta], \tag{2.1}$$

where $f : \Omega \times [t_0, t_0+\eta] \to \mathbb{R}^n$, $\Omega \subset C([-r, 0], \mathbb{R}^n)$ and x_t is given by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$, for every $t \in [t_0, t_0+\eta]$. The integral form of (2.1) is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}s, \quad t \in [t_0, t_0+\eta],$$

where the integral is in the sense of Henstock–Kurzweil.

The theory of measure functional differential equations deals with problems as

$$\mathrm{D}x = f(x_t, t)\mathrm{D}g,$$

where $\mathrm{D}x$ and $\mathrm{D}g$ denote the distributional derivatives in the sense of L. Schwartz of the functions x and g, respectively. The integral form is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\eta], \tag{2.2}$$

where we consider the integral on the right-hand side to be Henstock–Kurzweil–Stieltjes (we write H–K–S, for short) integrable with respect to a nondecreasing function g. See [6,7].

We assume that g is a left-continuous and nondecreasing function and consider the possibility of adding impulses at preassigned times $t_1, \dots, t_m$, where $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \eta$. For every $k \in \{1, \dots, m\}$, the impulse at t_k is described by the operator $I_k : \mathbb{R}^n \to \mathbb{R}^n$. In other words, the solution x should satisfy $\Delta^+ x(t_k) = I_k(x(t_k))$, where $\Delta^+ x(t_k) = x(t_k+) - x(t_k)$ and $x(t_k+) = \lim_{t\to t_k+} x(t)$. This leads us to the problem