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Perturbations on the antidiagonals of Hankel matrices

K. Castillo [a], D.K. Dimitrov [a], L.E. Garza [b,*], F.R. Rafaeli [a]

[a] Departamento de Matemática Aplicada, Universidade Estadual Paulista – IBILCE/UNESP, Rua Cristóvão Colombo, 2265 – Jardim Nazareth, 15054 000 São José do Rio Preto, SP, Brazil

[b] Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Mexico

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ABSTRACT

Given a strongly regular Hankel matrix, and its associated sequence of moments which defines a quasi-definite moment linear functional, we study the perturbation of a fixed moment, i.e., a perturbation of one antidiagonal of the Hankel matrix. We define a linear functional whose action results in such a perturbation and establish necessary and sufficient conditions in order to preserve the quasi-definite character. A relation between the corresponding sequences of orthogonal polynomials is obtained, as well as the asymptotic behavior of their zeros. We also study the invariance of the Laguerre-Hahn class of linear functionals under such perturbation, and determine its relation with the so-called canonical linear spectral transformations.

1. Introduction

1.1. Hankel matrices and orthogonal polynomials

Given a sequence of complex numbers $\{\mu_n\}_{n\geqslant 0}$, one can define a linear functional $\mathcal{M}$ in the linear space of polynomials with complex coefficients $\mathbb{P}$ such that

$$\langle \mathcal{M}, x^n\rangle = \mu_n. \tag{1}$$

In the literature (see [9,17], among others), $\mathcal{M}$ is said to be a moment linear functional, and the complex numbers $\{\mu_n\}_{n\geqslant 0}$ are called the moments associated with $\mathcal{M}$. The semi-infinite matrix

$$\mathbf{H} = \left[\langle \mathcal{M}, x^{i+j}\rangle\right]_{i,j=0,1,\dots} = \left[\mu_{i+j}\right]_{i,j=0,1,\dots} = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} \tag{2}$$

is the Gram matrix associated with the bilinear form of the linear functional (1) in terms of the canonical basis $\{x^n\}_{n\geqslant 0}$ of $\mathbb{P}$. If there exist a family of monic polynomials such that $\deg(P_n) = n$ and

$$\langle \mathcal{M}, P_n(x)P_m(x)\rangle = \gamma_n^{-2}\delta_{n,m}, \quad \gamma_n \neq 0, \quad n,m \geqslant 0,$$

* Corresponding author.
E-mail addresses: kcastill@math.uc3m.es (K. Castillo), dimitrov@ibilce.unesp.br (D.K. Dimitrov), garzaleg@gmail.com (L.E. Garza), rafaeli@ibilce.unesp.br (F.R. Rafaeli).

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where $\delta_{n,m}$ is the Kronecker delta, then $\{P_n\}_{n\geqslant 0}$ is called the monic orthogonal polynomials sequence (MOPS) associated with $\mathcal{M}$.

The Hankel matrices and their determinants play an important role in the study of moment functionals. The linear functional (1) is called quasi-definite if the moments matrix is strongly regular or, equivalently, the determinants of the principal leading submatrices $\mathbf{H}_n$ of order $(n+1)\times(n+1)$ are all different from 0. In this case, there exists a unique MOPS associated with $\mathcal{M}$.

On the other hand, a linear functional $\mathcal{M}$ is called positive definite if and only if its moments are all real and $\det \mathbf{H}_n > 0, n \geqslant 0$. In such a case, there exist a unique sequence of real polynomials $\{p_n\}_{n\geqslant 0}$ orthonormal with respect to $\mathcal{M}$, i.e., the following condition holds

$$\langle \mathcal{M}, p_n(x)p_m(x)\rangle = \delta_{n,m},$$

where

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + (\text{lower degree terms}), \quad \gamma_n > 0, \quad n \geqslant 0.$$

From the Riesz representation theorem, we know that every positive definite linear functional $\mathcal{M}$ has an integral representation (not necessarily unique)

$$\langle \mathcal{M}, x^n\rangle = \int_I x^n d\mu(x),$$

where μ denotes a nontrivial measure supported on some infinite subset I of the real line.

One of the most important characteristics of orthonormal polynomials on the real line is the fact that any three consecutive polynomials are connected by the simple recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{3}$$

with initial conditions $p_{-1} \equiv 0, \ p_0 \equiv \mu_0^{-1/2}$, and recurrence coefficients given by

$$a_n = \int_I xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0,$$
$$b_n = \int_I xp_n^2(x)d\mu(x) = \frac{\delta_n}{\gamma_n} - \frac{\delta_{n+1}}{\gamma_{n+1}}.$$

There are explicit formulae for orthonormal polynomials in terms of the determinants of the corresponding Hankel matrix. The n-th degree orthonormal polynomial is given by the Heine's formula

$$p_n(x) = \frac{1}{\sqrt{\det \mathbf{H}_n \det \mathbf{H}_{n-1}}}\begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

while its leading coefficient is given by a ratio of two Hankel determinants

$$\gamma_n = \sqrt{\frac{\det \mathbf{H}_{n-1}}{\det \mathbf{H}_n}}.$$

The n-th order reproducing kernel associated with $\{p_n\}_{n\geqslant 0}$ is defined by

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y), \quad n \geqslant 0.$$

The name comes from the fact that, for any polynomial q_n of degree at most n, we have

$$q_n(y) = \int_I q_n(x)K_n(x,y)d\mu(x).$$

The reproducing kernel can be represented in a simple way in terms of the polynomials p_n and p_{n+1} using the Christoffel-Darboux formula (see [9,17], among others)

$$K_n(x,y) = a_{n+1}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}, \quad x \neq y,$$

which can be deduced in a straightforward way from the three-term recurrence relation (3). We will denote by $K_n^{(i,j)}(x,y)$ the *i-th* (resp. *j-th*) partial derivative of $K_n(x,y)$ with respect to the variable x (resp. y). For the quasi-definite case, the reproducing kernel is defined as