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Inequalities for Integro-differential equations involving derivatives of order between zero and two

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ABSTRACT

We obtain bounds for integro-differential inequalities involving derivatives of orders between 0 and 2. We give applications and examples demonstrating the use of these bounds in analyzing the existence and asymptotic behavior of solutions for a class of singular fractional differential equations.

1. Introduction

We consider inequalities of the form

$$\left|D_{a^+}^{\alpha}u(t)\right| \leqslant a(t) + b(t)\int_a^t c(s)\left(\sum_{j=0}^{k}\left|D_{a^+}^{\beta_j}u(s)\right|\right)^n ds, \quad t > a,$$

$$\left|D_{a^+}^{\alpha}u(t)\right| \leqslant a(t) + \int_a^t c(s)\sum_{i=0}^{m}\left|D_{a^+}^{\gamma_i}u(s)\right|\sum_{j=0}^{k}\left|D_{a^+}^{\beta_j}u(s)\right|ds, \quad t > a,$$

where $1 < \alpha < 2, 0 \leqslant \beta_0 < \beta_1 < \ldots < \beta_k < \alpha, 0 \leqslant \gamma_0 < \gamma_1 < \ldots < \gamma_m < \alpha$, where $D_{a^+}^{\alpha}$ is the Riemann–Liouville derivative. The coefficients $a(t), b(t), c(t)$ are singular but integrable functions at the lower end of the interval of definition. These inequalities arise naturally when investigating nonlinear differential equations of fractional order of the form

$$D_{a^+}^{\alpha}u(t) = f\left(t, \left\{D_{a^+}^{\beta_j}u(t)\right\}_{j=0}^{k}, \left\{D_{a^+}^{\gamma_i}u(t)\right\}_{i=0}^{m}\right).$$

These inequalities yield bounds that can be used to investigate the qualitative behavior of the solutions. In particular, the bounds can be used to guarantee the non-blow-up and the asymptotic behavior for large t of the solutions.

In [2] inequalities involving different types of fractional derivatives were considered. In [1,4–6] bounds for inequalities involving derivatives of orders between 0 and 1 are obtained. Unlike in [4,5], in [1,6] the coefficients can be singular at the lower end of the intervals. In [5,6] the bounds on the largest derivative need not be as regular as in [1,4], and thus different classical inequalities are used. All these results may be seen as generalizations and extensions of analogous ones for derivatives of integer order found, for example, in [3,9].

In this paper we extend the results in [1] to derivatives of order between 0 and 2. This extension gives rise to an extra term in the composition identity of integrals with derivatives. This term in general is not integrable. However, as in [1], we can reduce the fractional inequality to a classical one and obtain the bounds.

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In Section 2 we introduce some preliminaries. In Section 3 we present our results and their proofs. Section 4 is devoted to some applications.

2. Preliminaries

We present the necessary definitions and results used in this work. For more details we refer the reader to [7–11].

We denote by $L_p,\ 1 \leqslant p \leqslant \infty$, the Lebesgue spaces, and by $AC[a,b]$ the space of all absolutely continuous functions on $[a,b]$, and by $AC^n[a,b]$, where $n = 1,2,\ldots$, the space of functions f which have continuous derivative up to order $n-1$ on $[a,b]$ with $f^{(n-1)} \in AC[a,b], -\infty < a < b < \infty$. We use the terms non-increasing and non-decreasing to refer to monotonic functions only.

Definition 1. Let $f \in L_1(a,b)$, the integral

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a, \quad \alpha > 0$$

is called the Riemann–Liouville fractional integral of order α of the function f. Here Γ is the Euler's gamma function.

Definition 2. The expression

$$D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad n = -[-\alpha],\ x > a,\ \alpha > 0$$

is called the Riemann–Liouville fractional derivative of order α of the function f, where $[\alpha]$ is the integer part of α.

We use f_α to denote $I^{\alpha}_{a^+} f$ and we set $I^0_{a^+} f = D^0_{a^+} f = f$.

Definition 3. Let $\alpha > 0$. A function $f \in L_1(a,b)$ is said to have a summable fractional derivative $D^{\alpha}_{a^+} f$ on (a,b) if $I^{n-\alpha}_{a^+} f \in AC^n[a,b], n = -[-\alpha]$.

Definition 4. We define the space $I^{\alpha}_{a^+}(L_p(a,b)), \alpha > 0, 1 \leqslant p < \infty$ to be the space of all functions f such that $f = I^{\alpha}_{a^+} \varphi$ for some $\varphi \in L_p(a,b)$.

Theorem 5. *$f \in I^{\alpha}_{a^+}(L_1(a,b)), \alpha > 0$, if and only if $f_{n-\alpha} \in AC^n[a,b], n = -[-\alpha]$, and*

$$f^{(k)}_{n-\alpha}(a) = 0, \quad k = 0,1,2,\ldots,n-1.$$

See [11, Theorem 2.3, p. 43].

Proposition 6. *If f has a summable fractional derivative $D^{\beta}_{a^+} f, 1 < \beta < 2$, on (a,b), then for $\alpha \geqslant \beta$, we have the*

$$I^{\alpha}_{a^+} D^{\beta}_{a^+} f(t) = f_{\alpha-\beta}(t) - \frac{D^{\beta-1} f(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1} - \frac{f_{2-\beta}(a)}{\Gamma(\alpha-1)} (t-a)^{\alpha-2},$$

almost everywhere in (a,b). See [11, p. 48].

Corollary 7. *If $f \in L_1(a,b)$ has a summable fractional derivative $D^{\alpha}_{a^+} f, 1 < \alpha < 2$, on (a,b), then for $0 \leqslant \beta < \alpha$, we have*

$$D^{\beta}_{a^+} f(t) = I^{\alpha-\beta}_{a^+} D^{\alpha}_{a^+} f(t) + \frac{D^{\alpha-1} f(a)}{\Gamma(\alpha-\beta)} (t-a)^{\alpha-\beta-1} + \frac{f_{2-\alpha}(a)}{\Gamma(\alpha-\beta-1)} (t-a)^{\alpha-\beta-2}.$$

Proof. In Proposition 6, replace β by α, and replace α by $\alpha - \beta$. □

Remark 1. If $\alpha - \beta > 1$ or $f_{2-\alpha}(a) = 0$ in Corollary 7 then $D^{\beta}_{a^+} f \in L_1(a,b)$.

Lemma 8. *Let v, f, g and k be non-negative continuous functions on $[a,b]$. Let ω be a continuous, non-negative and non-decreasing function on $[0,\infty)$, with $\omega(0) = 0$ and $\omega(u) > 0$ for $u > 0$. Let $F(t) = \max_{0 \leqslant s \leqslant t} f(s)$ and $G(t) = \max_{0 \leqslant s \leqslant t} g(s)$. If*

$$v(t) \leqslant f(t) + g(t) \int_a^t k(s) \omega(v(s)) ds, \quad t \in [a,b],$$

then