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On a system of difference equations with maximum

N. Fotiades, G. Papaschinopoulos *

School of Engineering, Democritus University of Thrace, Xanthi 67100, Greece

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ABSTRACT

In this paper we study the periodic character of the solutions of the system of difference equations with maximum

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\},$$
$$y_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\},$$

where A, B are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers. We prove that every solution is eventually periodic.

1. Introduction

In [1] the authors proved that every solution of the difference equation

$$y_{n+1} = \max\left\{D, \frac{y_n}{y_{n-1}}\right\} \tag{1.1}$$

is eventually periodic. That is there exist $T, n_0 \in \mathbb{N}$ such that $y_{n+T} = y_n$ for every $n \geqslant n_0$.

Using this result and the transformation $y_{n+1} = \frac{x_{n+1}x_n}{B}$ the authors proved that every solution of the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}, \quad A, B > 0$$

is eventually periodic.

Two important papers on a generalization on Eq. (1.1) are: [2,3].

Moreover in [4] the authors study the periodicity of the positive solutions of the system

$$\begin{aligned} z_{n+1} &= \max\left\{\frac{B}{w_{n+1-k}}, \frac{D}{w_{n+1-m}}\right\}, \\ w_{n+1} &= \max\left\{\frac{C}{z_{n+1-k}}, \frac{E}{z_{n+1-m}}\right\}, \end{aligned} \tag{1.2}$$

where k, m are positive integers.

Motivated by Eq. (1.1) we study the system

$$\begin{aligned} x_{n+1} &= \max\left\{A, \frac{y_n}{x_{n-1}}\right\}, \\ y_{n+1} &= \max\left\{B, \frac{x_n}{y_{n-1}}\right\}, \end{aligned} \tag{1.3}$$

* Corresponding author.
E-mail addresses: nikos.fotiades@gmail.com (N. Fotiades), gpapas@env.duth.gr (G. Papaschinopoulos).

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$n \geqslant 0$ where A, B are positive numbers and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers. We prove that every solution of (1.3) is eventually periodic. We note that if $A = B = D$ and $x_{-1} = y_{-1}, x_0 = y_0$ then system (1.3) reduces to the Eq. (1.1). To study our results we examine three cases:

$$(a)\ 1 \leqslant A \leqslant B, \quad (b)\ A < 1 \leqslant B, \quad (c)\ A \leqslant B < 1 \tag{1.4}$$

and the similar cases:

$$(a)\ 1 \leqslant B \leqslant A, \quad (b)\ B < 1 \leqslant A, \quad (c)\ B \leqslant A < 1. \tag{1.5}$$

We note that if $x_{n+1} = \kappa z_{n+1} w_n$ and $y_{n+1} = \lambda z_n w_{n+1}$ where $\kappa, \lambda \in \mathbb{R}$ then from system (1.3) we take the system

$$\begin{aligned} z_{n+1} &= \max\left\{\frac{A/\kappa}{w_n}, \frac{\lambda/\kappa^2}{w_{n-2}}\right\}, \\ w_{n+1} &= \max\left\{\frac{B/\lambda}{z_n}, \frac{\kappa/\lambda^2}{z_{n-2}}\right\}, \end{aligned} \tag{1.6}$$

which results from the system (1.2) with $k = 1, m = 3$, $B = \frac{A}{\kappa}, C = \frac{B}{\lambda}, D = \frac{\lambda}{\kappa^2}$, $E = \frac{\kappa}{\lambda^2}$.

As we show below only the third case of (1.4) and (1.5) follows immediately from the study of system (1.6).

The max operator arises naturally in certain models in automatic control theory (see [5]).

For some other papers with difference equations with maximum you can see [1–27].

2. Main Results

In the following we use a Lemma which has been proved in [4] (see Lemma 1). For readers convenience we state it here without its proof.

Lemma 2.1. *Consider the system of difference Eq. (1.2) where B, C, D, E are positive real constants, k, m are positive integers and the initial values $z_i, w_i, i = -d, -d+1, \ldots, -1$, $d = \max\{k, m\}$, are positive real numbers. Then the following statements are true:*

(i) If either $B = C$, $B \geqslant D \geqslant E$, B, C, D, E are not all equal or $B = C$, $B \geqslant E \geqslant D$, B, C, D, E are not all equal then every positive solution of the system (1.2) is eventually periodic with period $2k$.
(ii) If either $D = E$, $D \geqslant B \geqslant C$, B, C, D, E are not all equal or $D = E$, $D \geqslant C \geqslant B$, B, C, D, E are not all equal then every positive solution of the system (1.2) is eventually periodic with period $2m$.

Proposition 2.1. *Let (x_n, y_n) be a solution of system (1.3). The following statements are true:*

(I) *If $1 \leqslant A \leqslant B$ then*
  (i) $y_n = B$ for every $n \geqslant 4$.
  (ii) $x_n = \begin{cases} A, & \text{for every } n \geqslant 7 \quad \text{if } B \leqslant A^2, \\ x_{n+4}, & \text{for every } n \geqslant 5 \quad \text{if } B > A^2. \end{cases}$

(II) *If $1 \leqslant B \leqslant A$ then*
  (i) $x_n = A$ for every $n \geqslant 4$.
  (ii) $y_n = \begin{cases} B, & \text{for every } n \geqslant 7 \quad \text{if } A \leqslant B^2, \\ y_{n+4}, & \text{for every } n \geqslant 5 \quad \text{if } A > B^2. \end{cases}$

Proof. (I) (i) From the recurrence Eq. (1.3) it follows that $x_n \geqslant A$ and $y_n \geqslant B$ for every $n \geqslant 1$. Assume that $y_n > B$ for some $n \geqslant 4$. Then from (1.3) we get

$$y_n = \frac{x_{n-1}}{y_{n-2}} = \frac{1}{y_{n-2}} \max\left\{A, \frac{y_{n-2}}{x_{n-3}}\right\} = \max\left\{\frac{A}{y_{n-2}}, \frac{1}{x_{n-3}}\right\}.$$

We have $\frac{A}{y_{n-2}} \leqslant \frac{A}{B} \leqslant 1 \leqslant B$ and $\frac{1}{x_{n-3}} \leqslant \frac{1}{A} \leqslant 1 \leqslant B$. So $y_n \leqslant B$ which contradicts the assumption $y_n > B$. Thus $y_n = B$ for every $n \geqslant 4$.

(ii) We examine two cases:

1st Case. Assume that $B \leqslant A^2$.

Let $n \geqslant 7$. Then from (1.3) and since $y_n = B, n \geqslant 4$ we have

$$x_n = \max\left\{A, \frac{B}{x_{n-2}}\right\} = \max\left\{A, \frac{B}{\max\left\{A, \frac{B}{x_{n-4}}\right\}}\right\} = \max\left\{A, \min\left\{\frac{B}{A}, x_{n-4}\right\}\right\}.$$

We have $x_{n-4} \geqslant A \geqslant \frac{B}{A}$ since $B \leqslant A^2$. So $x_n = \max\left\{A, \frac{B}{A}\right\} = A$.