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Application of the restrained optimal perturbation method to study the backward heat conduction problem

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ABSTRACT

In this paper, a restrained optimal perturbation method is firstly proposed to solve the backward heat conduction problem, the initial temperature distribution will be identified from the overspecified data, a regularization term is introduced in the objective functional for overcoming the ill-posedness of this problem, spectral projected gradient algorithm is used to solve the optimal problem, and we give the sensitivity analysis of the initial value. The results of numerical experiments are also presented.

1. Introduction

In this paper, we consider the following heat conduction problem governed by

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\psi(u),\quad 0\leqslant x\leqslant \pi,\ 0<t\leqslant T, \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t)=0,\quad 0<t\leqslant T, \tag{2}$$

$$\frac{\partial u}{\partial x}(\pi,t)=0,\quad 0<t\leqslant T, \tag{3}$$

$$u(x,0)=\varphi(x),\quad 0\leqslant x\leqslant \pi, \tag{4}$$

where the nonlinear term $\psi(u)$ may be interpreted as a heat or material source when u represents temperature or concentration respectively, while in a chemical or biochemical application, $\psi(u)$ is interpreted as a reaction term.

Determination of the unknown source term in this problem has been discussed by many authors [1–5], in all of these works, the initial condition and boundary conditions are considered as known functions, while Shidfar et al. have considered the identification of heat flux histories at $x=0$ in this problem [6], and there are only several papers with regard to the identification of the initial temperature in the linear source term or without source term about this problem [7–10].

In this paper, our target is to determine the initial temperature $\varphi(x)$ in the above problem from the final measurement, and we give the overspecified data at time $t=T$ as follows

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$$u(x,T) = g(x), \quad 0 \leqslant x \leqslant \pi, \tag{5}$$

in which $g(x)$ is considered as a known function.

This paper is organized as follows. In the next section, we will give a brief description of the restrained optimal perturbation method, and the spectral projected gradient algorithm is used to compute the optimal perturbation. In Section 3, we give the sensitivity analysis of the initial problem. In Section 4, we will use the finite difference approximation to discrete the space derivative, and transform the heat conduction problem into the system of ordinary differential equations (ODEs), then we apply restrained optimal perturbation method for solving the initial temperature distribution. In Section 5, numerical experiments will be given to investigate the applications of this method. The conclusion and discussion are presented in the last section.

2. A restrained optimal perturbation method

Conditional nonlinear optimal perturbation (CNOP) is proposed by Mu et al. (2003) [11], and has been utilized to study ENSO, the ocean's thermohaline circulation problem, a theoretical grassland ecosystem, etc. [12–20]. Their work shows that CNOP is an useful tool in the studies of predictability, sensitivity and nonlinear stability analysis.

Inspired by them, we propose a new method to solve the inverse problem by calculating the restrained optimal perturbation, i.e., restrained optimal perturbation method. Now let us give a brief introduction to this method. Assuming that the mathematical model is as following:

$$\frac{\partial U}{\partial t} = F(U), \tag{6}$$

$$U|_{t=0} = U_0, \tag{7}$$

where $U(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)), F$ is nonlinear (or linear) operator, and $(x,t) \in \Omega \times [0,T]$, Ω is a domain in R^n and $T < +\infty, U_0$ is the initial estimate, which can be obtained from the long experience value. Supposing R is the propagator from 0 to time T(in fact, it is a solution operator), then, for fixed $T > 0$, the solution $U(x,T) = R(U_0)(T)$ is well-defined. Let $U(x,t)$ and $U(x,t) + u(x,t)$ be the solutions of problem (6) and (7) with initial estimate value U_0 and $U_0 + u_0$ respectively, where u_0 is the initial perturbation. We have

$$U(T) = R(U_0)(T), \tag{8}$$

$$U(T) + u(T) = R(U_0 + u_0)(T). \tag{9}$$

So $u(T)$ describes the evolution of the initial perturbation u_0. The perturbation $u_{0\delta}$ is called the optimal perturbation, if and only if

$$J(u_{0\delta}) = \min_{u_0} J(u_0).$$

Owing to the inverse problem belong to the ill-posed problem, to overcome the difficulty of ill-posedness, a regularization term is introduced in the objective functional, so it can be written as

$$J(u_0) = \|R(U_0 + u_0)(T) - R(U_0)(T)\| + \lambda \|U_0\|, \tag{10}$$

in which $\lambda > 0$ is a regularizing parameter.

The optimal perturbation $u_{0\delta}$ obtained from the above must satisfies the condition

$$E = \|R(U_0 + u_{0\delta})(T) - G(T)\| \leqslant \delta, \tag{11}$$

such that the error E is sufficiently small, where $G(T)$ is discrete values of the observational data, if not, we will repeat this process until the $u_{0\delta}$ occurs.

The above constrained optimization minimum value problem can be transformed into the following Lagrangian problem

$$J(u_{0\delta}) = \min_{u_0} J(u_0) + \frac{\mu}{2} \|R(U_0 + u_0)(T) - G(T)\|,$$

where μ is Lagrange multiplier.

In this paper, spectral projected gradient (SPG) algorithm is adopted to solve the above Lagrangian problem, the detailed description of this algorithms can be found in Birgin et al. (Refs. [21,22]).

3. The sensitivity analysis of the initial problem

Here, we introduce the Hilbert space $H = L_2(\Omega)$ in which the scalar product is defined as

$$(u, v) = \int_\Omega u(x) v(x) dx. \tag{12}$$