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Uniform boundedness and convergence of global solutions to a strongly-coupled parabolic system with three competitive species

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ABSTRACT

Uniform boundedness and convergence of global solutions are proved for strongly-coupled parabolic systems with cross-diffusions dominated by self-diffusions in population dynamics, Gagliardo–Nirenberg inequalities are used in the estimates of solutions in order to establish W_2^1-bounds uniform in time.

1. Introduction and basic proposition

In an attempt to model spatial segregation phenomena between two competing species, Shigesada et al. [1] proposed the following quasilinear parabolic system (P_0) in 1979:

$$\begin{cases} u_t = \triangle[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + (a_1 - b_1u - c_1v)u, & (x,t) \in \Omega \times (0,\infty) \\ v_t = \triangle[(d_2 + \alpha_{12}u + \alpha_{22}v)v] + (a_2 - b_2u - c_2v)v, & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) \geqslant 0,\ v(x,0) = v_0(x) \geqslant 0 & x \in \overline{\Omega} \end{cases} \qquad (P_0)$$

so far the existence of nonnegative global solutions for system (P_0) has been proved extensively in one or two dimension [2–10]. Shim [9] established the global uniform boundedness and convergence for system (P_0) with $n = 1$ under the condition $0 < \alpha_{21} < 8\alpha_{11},\ 0 < \alpha_{12} < 8\alpha_{22}$. In recent years more and more attention have been given to system (P_0) with other types of reaction term and some generalized three-species [11–13] as the following system (P):

$$\begin{cases} u_t = \triangle[(E_1 + \alpha_{11}u + \alpha_{12}v + \alpha_{13}w)u] + (a_1 - b_1u - c_1v - d_1w)u, & (x,t) \in \Omega \times (0,\infty) \\ v_t = \triangle[(E_2 + \alpha_{12}u + \alpha_{22}v + \alpha_{23}w)v] + (a_2 - b_2u - c_2v - d_2w)v, & (x,t) \in \Omega \times (0,\infty) \\ w_t = \triangle[(E_3 + \alpha_{13}u + \alpha_{23}v + \alpha_{33}w)w] + (a_3 - b_3u - c_3v - d_3w)w, & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) \geqslant 0,\ v(x,0) = v_0(x) \geqslant 0,\ w(x,0) = w_0(x) \geqslant 0, & x \in \overline{\Omega} \end{cases} \qquad (\mathrm{P})$$

In order to prove uniform boundedness and convergence of global solutions to the above system (P),we consider the following case for system (P):(A) $2\alpha_{ii}\alpha_{ji} > \alpha_{ij}^2 (i \neq j, i,j = 1,2,3)$.

Where $\Omega = [0,1],\ u_0(x),\ v_0(x),\ w_0(x) \in W_2^1[0,1]$. In system (P) u, v, w are nonnegative functions which represent the population densities of three competing species. $\alpha_{ij},\ E_i;\ d_i;\ a_i;\ b_i > 0 (i,j = 1,2,3)$. $E_1,\ E_2,\ E_3$ are the diffusion rates of the three species, respectively. $a_1,\ a_2,\ a_3$ denote the intrinsic growth rates, $b_1,\ c_2,\ d_3$ account for intra-specific competitions,

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$b_2,\ d_3,\ c_1,\ d_3,\ d_1,\ d_2$ are the coefficients for inter-specific competitions, $\alpha_{11},\ \alpha_{22},\ \alpha_{33}$ denote self-diffusion, and $\alpha_{ij}(i \neq j, i, j = 1, 2, 3)$ are cross-diffusion pressures. By adopting the coefficients $\alpha_{ij}(i,\ i, j = 1, 2, 3)$, system (P) takes into account the pressures created by mutually competing species.

To describe results on system (P) we use the following notation throughout this paper.

Notation. Let Ω be a region in R^n. The norm in $L_p(\Omega)$ is denoted by $|.|_{L_p(\Omega)},\ 1 \leqslant p \leqslant \infty$. The usual Sobolev spaces of real valued functions in Ω with exponent $k \geqslant 0$ are denoted by $W^{k,p}(\Omega),\ 1 \leqslant p \leqslant \infty$. And $||.||_{k,p}$ represents the norm in Sobolev spaces $W^{k,p}(\Omega)$. We shall use the simplified notation $||.||_{k,p}$ for $W^{k,p}(\Omega)$ and $|.|_p$ for $L^p(\Omega)$.

The local existence of solutions to system (P) was established by Amamn [14–16]. According to his results system (P) has a unique nonnegative solution $(u(x,t), v(x,t), w(x,t)) \in C([0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\Omega))$, where $T \in (0,\infty]$ is the maximal existence time for the solution. The following results is due to Amamn [15].

Theorem 1.1. *If $u_0(x),\ v_0(x),\ w_0(x) \in W_p^1(\Omega),\ \Omega \subset R^n$ is bounded, $p > n$. System (P) possesses a unique solution: $(u(x,t), v(x,t), w(x,t)) \in C([0,T), W_p^1(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,T))$ for $\forall 0 \leqslant t < T$, where $p > n,\ 0 < T < \infty$. If the solutions satisfy the estimates*

$$\sup_{0<t<T} \|u(.,t)\|_{W_p^1(\Omega)} < \infty, \quad \sup_{0<t<T} \|v(.,t)\|_{W_p^1(\Omega)} < \infty, \quad \sup_{0<t<T} \|w(.,t)\|_{W_p^1(\Omega)} < \infty.$$

then $T = \infty$.

Theorem 1.2 (*Gagliardo–Nirenberg inequalities*). *Let $\Omega \in R^n$ be a bounded domain with $\partial\Omega \in C^m$. For every function $u \in W^{m,r}(\Omega)(1 \leqslant q, r \leqslant \infty), D^j u(0 \leqslant j < m)$ satisfies the inequalities:*

$$|D^j u|_p \leqslant C\left(|D^m u|_r^a |u|_q^{1-a} + |u|_q\right) \tag{1.1}$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$.

for all a in the interval $\frac{j}{m} \leqslant a < 1$, *provided one of the following three conditions:*

(1) $r \leqslant q$,
(2) $0 < \frac{n(r-q)}{mrq} < 1$,
(3) $\frac{n(r-q)}{mrq} = 1$ and $m - \frac{n}{q}$ is not a nonnegative integer.

(The positive constant C depends only on $n,\ m,\ j,\ q,\ r,\ a$.)

Proof. We refer the readers to Theorem 10.1 in Part 1 of Friedman [17] for the proof of this well-known calculus inequality. □

Corollary 1.3. *There exists positive constants $c,\ c^*,\ c^{**}$ such that for every function u in $H^1[0,1]$:*

$$|u|_2 \leqslant c\left(|u_x|_2^{\frac{1}{3}} |u|_1^{\frac{2}{3}} + |u|_1\right) \tag{1.2}$$

$$|u|_4 \leqslant c^*\left(|u_x|_2^{\frac{1}{2}} |u|_1^{\frac{1}{2}} + |u|_1\right) \tag{1.3}$$

$$|u|_5 \leqslant c^{**}\left(|u_x|_2^{\frac{3}{10}} |u|_2^{\frac{7}{10}} + |u|_2\right) \tag{1.4}$$

$$|u|_\infty \leqslant c^{***}\left(|u_x|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}} + |u|_2\right) \tag{1.5}$$

Proof. n = 1, m = 1, j = 0, r = 2, q = 1 satisfy condition (2) in Theorem 1.2, thus (1.2) and (1.3) are correct. n = 1, m = 1, j = 0, r = 2, q = 2 satisfy condition (1) in Theorem 1.2, thus (1.4) and (1.5) are correct. □

Remark. From (1.5) we have a conclusion that

$$W_2^1([0,1]) \hookrightarrow C([0,1]) \tag{1.6}$$

Lemma 1.4. *For every function $u \in W^{2,2}([0,1])$ with $u_x(0) = u_x(1) = 0$.*

$$|u_x|_2 \leqslant |u_{xx}|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}} \tag{1.7}$$