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On a relationship between Chebyshev polynomials and Toeplitz determinants

Mohamed Elouafi

Classes Préparatoites aux Grandes Ecoles d'Ingénieurs, Lycée My Alhassan, Tangier, Morocco

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ABSTRACT

Explicit formulas are given for the determinants of a band symmetric Toeplitz matrix T_n with bandwidth $2r+1$. The formulas involve $r \times r$ determinants whose entries are the values of Chebyshev polynomials on the zeros of a certain rth degree q which is independent of n.

1. Introduction

Toeplitz matrix is a matrix which has constant values along negative-sloping diagonals, i.e., a matrix of the form

$$\mathbf{T}_n = \begin{pmatrix} a_0 & a_1 & \cdots & a_r & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \ddots & \cdots & \vdots \\ \vdots & a_{-1} & a_0 & a_1 & \ddots & a_r \\ a_{-r} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_0 & a_1 \\ a_{-(n-1)} & \cdots & a_{-r} & \cdots & a_{-1} & a_0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{C}).$$

In this article, we are concerned with banded symmetric Toeplitz matrices $\mathbf{T}_n$ with bandwidth $2r+1$, meaning that $a_k = 0$ if $|k| > r, a_k = a_{-k}$ and $a_r \neq 0$. We silently assume that n is large in comparison with $2r+1$. Band Toeplitz matrices arise in many different theoretical and applicative fields. Their determinants, called Toeplitz determinants, have important applications in many topics of theoretical physics: statistical physics, random-matrix theory, and have been studied by several authors [1].

Since $\mathbf{T}_n$ is a centrosymmetric matrix, then det $(\mathbf{T}_n)$ is a product of two factors [6] which we express using Chebyshev polynomials of the first, second, third, and fourth kind, which will be denoted by T_n, U_n, V_n and W_n, respectively [4] and the zeros of $f(z) = \sum_{k=-r}^{r} a_k z^k$. This is our main result:

Theorem 1. *Let $\zeta_j, \frac{1}{\zeta_j}, j = 1, \ldots, r$ are the (distincts) zeros of the polynomial $g(z) = z^r f(z)$. Then we have for all $p \geqslant 1$:*

E-mail address: med3elouafi@gmail.com

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$$\det(\mathbf{T}_{2p}) = \frac{a_r^{2p}}{2^{r(r-1)}} \times \frac{\begin{vmatrix} V_p(\alpha_1) & V_p(\alpha_2) & \cdots & V_p(\alpha_r) \\ V_{p+1}(\alpha_1) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ V_{p+r-1}(\alpha_1) & V_{p+r-1}(\alpha_2) & \cdots & V_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leqslant i<j\leqslant r}(\alpha_j-\alpha_i)} \times \frac{\begin{vmatrix} W_p(\alpha_1) & W_p(\alpha_2) & \cdots & W_p(\alpha_r) \\ W_{p+1}(\alpha_1) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ W_{p+r-1}(\alpha_1) & W_{p+r-1}(\alpha_2) & \cdots & W_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leqslant i<j\leqslant r}(\alpha_j-\alpha_i)}$$

and

$$\det(\mathbf{T}_{2p+1}) = \frac{(-1)^r a_r^{2p+1}}{2^{r(r-2)}} \times \frac{\begin{vmatrix} U_p(\alpha_1) & U_p(\alpha_2) & \cdots & U_p(\alpha_r) \\ U_{p+1}(\alpha_1) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ U_{p+r-1}(\alpha_1) & U_{p+r-1}(\alpha_2) & \cdots & U_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leqslant i<j<r}(\alpha_j-\alpha_i)} \times \frac{\begin{vmatrix} T_{p+1}(\alpha_1) & T_{p+1}(\alpha_2) & \cdots & T_{p+1}(\alpha_r) \\ T_{p+2}(\alpha_1) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ T_{p+r}(\alpha_1) & T_{p+r}(\alpha_2) & \cdots & T_{p+r}(\alpha_r) \end{vmatrix}}{\prod_{1\leqslant i<j<r}(\alpha_j-\alpha_i)},$$

where $\alpha_k = \frac{1}{2}\left(\zeta_k + \frac{1}{\zeta_k}\right),\ k = 1,2,\ldots,r$ are the zeros of the polynomial

$$q(x) = a_0 + 2\sum_{k=1}^{r} a_k T_k(x).$$

2. Proof of the Theorem 1

We have $f(z) = f\left(\frac{1}{z}\right)$ and hence if ζ is a zero of f then so also is $\frac{1}{\zeta}$. Thus the zeros of f are $\zeta_j, \frac{1}{\zeta_j}, j = 1,\ldots,r$ assumed pairwise distinct.

We begin by recalling some useful properties of the Chebyshev polynomials $\{T_n\}, \{U_n\}, \{V_n\}$ and $\{W_n\}$ [4]. They all satisfy the same recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n = 1,2,\ldots$$

and the different initials conditions are

$$T_0(x) = U_0(x) = 1, \quad 2T_1(x) = U_1(x) = 2x$$

and

$$W_0(x) = V_0(x) = 1, \quad W_1(x) = V_1(x) + 2 = 2x + 1.$$

Moreover, we have

$$\begin{cases} T_n(\cos\theta) = \cos n\theta, & U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \\ V_n(\cos\theta) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}, & W_n(\cos\theta) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \end{cases} \tag{2.1}$$

from which it follows the relations

$$U_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{\zeta^{n+1} - \zeta^{-n-1}}{\zeta - \zeta^{-1}} \quad \text{and} \quad T_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{1}{2}\left(\zeta^n + \frac{1}{\zeta^n}\right)$$

for $\zeta \in \mathbb{C}$. We will divide the proof of the Theorem 1 into many lemmas:

Lemma 1. *We have:*

(1) *Let $j,k \in \mathbb{N}$. Then*

$$U_j(x)T_k(x) = \begin{cases} \frac{1}{2}U_{j+k}(x) + \frac{1}{2}U_{j-k}(x) & \text{if } k \leqslant j+1, \\ \frac{1}{2}U_{j+k}(x) - \frac{1}{2}U_{k-j-2}(x) & \text{if } j+1 < k. \end{cases}$$

(2) *Let $j,k \in \mathbb{N}$. Then*

$$V_j(x)W_k(x) = \begin{cases} U_{j+k}(x) + U_{j-k-1}(x) & \text{if } k \leqslant j, \\ U_{j+k}(x) - U_{k-j-1}(x) & \text{if } j < k. \end{cases}$$