Applied Mathematics and Computation 229 (2014) 151–158

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence and stability of Euler method for impulsive stochastic delay differential equations

Kaining Wu *, Xiaohua Ding

Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, China

ARTICLE INFO

Keywords:
Impulsive stochastic delay differential equations
Numerical method
Euler method
Convergence
Mean square exponential stability

ABSTRACT

This paper deals with the mean square convergence and mean square exponential stability of an Euler scheme for a linear impulsive stochastic delay differential equation (ISDDE). First, a method is presented to take the grid points of the numerical scheme. Based on this method, a fixed stepsize numerical scheme is provided. Based on the method of fixed stepsize grid points, an Euler method is given. The convergence of the Euler method is considered and it is shown the Euler scheme is of mean square convergence with order 1/2. Then the mean square exponential stability is studied. Using Lyapunov-like techniques, the sufficient conditions to guarantee the mean square exponential stability are obtained. The result shows that the mean square exponential stability may be reproduced by the Euler scheme for linear ISDDEs, under the restriction on the stepsize. At last, examples are given to illustrate our results.

1. Introduction

In recent years, impulsive stochastic delay differential equations (ISDDEs) have been studied by many authors and a lot of results have been reported, for example, see [1,2] and the references therein. Generally, the explicit solutions of the ISDDEs are difficult to be obtained, thus it is necessary to develop numerical methods for ISDDEs and study the properties of these numerical methods.

For impulsive differential equations, there are many results on the numerical methods. In 2007, Wu [3] first brought out an Euler method for a random impulsive differential equation using a variable stepsize method. In the same year, in [4], the stability of the Runge–Kutta methods was studied for a linear impulsive differential equation with constant coefficients, where the authors also adopted the variable stepsize numerical scheme. Then these authors considered the nonlinear case in [5]. In [6], the authors considered some numerical methods for impulsive differential equations, where the interval length of the impulsive moments are assumed to be equal. This assumption makes the authors can take a fixed stepsize numerical scheme. In [7] the authors considered the numerical stability and asymptotical stability of the implicit Euler method for a stiff impulsive differential equation in Banach space where the variable stepsize numerical scheme was adopted.

The study of the numerical method for stochastic delay differential equations has been done for many years and a lot of results were reported, see [8–12] and the references therein. For the impulsive delay differential equations, the authors presented a fixed stepsize Euler scheme and considered the convergence of this Euler method in [13]. Moreover, we should point out that, for ISDDEs, in [14], the authors studied the exponential stability of the Euler method with fixed stepsize method. However, just as the case in [6], in that paper, the assumption of the length's equality of each impulsive interval was imposed and the length of the impulsive interval was asked to be equal to the delay. Both the assumptions yield a fixed stepsize numerical scheme for ISDDEs, which also reduced the applications scale of the numerical scheme.

* Corresponding author.
E-mail address: kainingwu@163.com (K. Wu).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.12.041

In this paper, we generalize the method of [13] to the case of ISDDEs. First, we present a method to take the grid points and this method yields a fixed stepsize numerical method for ISDDEs. Based on this method, we give an Euler method for the ISDDEs. Then we consider the convergence of the Euler scheme and obtain the order of the convergence. The mean square exponential stability of the Euler method is also studied and the sufficient conditions are obtained to guarantee the stability by using the Lyapunov-like techniques. At last, numerical examples are given to illustrate our results.

For the convenience, we adopt the following standard notations:

$(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ is a complete probability space with a filtration $\{F_t\}_{t\geqslant slant0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, \ldots\}$. Given a positive integer m, define $N_{-m} = \{-m, -m+1, \ldots, 0\}$. $PC(\mathbb{J}; \mathbb{R}) = \{\varphi : \mathbb{J} \to \mathbb{R}, \varphi(s)$ is continuous for all but at most countable points $s \in \mathbb{J}$ and at these points $s \in \mathbb{J}, \varphi(s^+)$ and $\varphi(s^-)$ exist and $\varphi(s^+) = \varphi(s)\}$, where $\mathbb{J}$ is a subset of $\mathbb{R}$, $\varphi(s^+)$ and $\varphi(s^-)$ denote the right-hand and left-hand limits of the function $\varphi(s)$ at time s respectively. $PC^b_{F_0}([-\tau, 0]; \mathbb{R})$ denotes the family of all bounded F_0-measurable, PC-valued random variables. $\|\phi\| = \sup_{-\tau\leqslant\theta\leqslant 0}\{|\phi(\theta)|\}$.

2. Preliminaries

In this paper, we will consider the following system:

$$\begin{cases} dx(t) &= [ax(t) + bx(t-\tau)]dt + cx(t)dW(t), \quad t \geqslant 0, \quad t \neq t_k, \\ x(t_k) &= \beta_k x(t_k^-), \end{cases} \tag{2.1}$$

where a, b, c, β_k are constants, $\tau > 0$ is the delay, $x(t_k^-) = \lim_{t\to t_k^-} x(t)$. The impulsive moments $\{t_k\}$ satisfy: $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$. $W(t)$ is an 1-dimension Brownian motion.

We impose the following initial data for system (2.1):

$$x(s) = \phi(s), \quad s \in [-\tau, 0], \tag{2.2}$$

where $\phi \in PC^b_{F_0}([-\tau, 0], \mathbb{R})$.

For the sake of simplicity, we do not consider the more general system:

$$\begin{cases} dx(t) &= [ax(t) + bx(t-\tau)]dt + [cx(t) + dx(t-\tau)]dW(t), \quad t \geqslant 0, \quad t \neq t_k, \\ x(t_k) &= \beta_k x(t_k^-), \end{cases} \tag{2.3}$$

there is no more essential difficulty to study the system (2.1) than to study system (2.3) except the complexity.

Now we state a lemma for system (2.1), it will be used in the sequel.

Lemma 2.1. *For any given positive constant T, there exists an $M > 0$, such that the solution of system (2.1) with initial data (2.2) satisfies:*

$$E|x(t, \phi)|^2 < M, -\tau \leqslant t \leqslant T.$$

Proof. From the property of the impulsive moments, we know that, for a given $T > 0$, there exists a natural number N such that $t_N \leqslant T < t_{N+1}$. By virtue of the exponential estimate theorem in [15] and the induction, we can get the required result easily. □

To end this section, we give a lemma on the mean square exponential stability of system (2.1). It can be obtained directly by the results of [1].

Lemma 2.2. *If the coefficients of system (2.1) satisfy:*

$$2a + 2|b| + c^2 < 0, \quad \beta_k^2 \geqslant 1 \quad \text{and} \quad \sum_{k=1}^{\infty}(\beta_k^2 - 1) < \infty, \tag{2.4}$$

then the trivial solution of system (2.1) is mean square exponentially stable.

3. Convergence of Euler method

In this section, we first present a method to get the grid points for numerical scheme, which is a fixed stepsize method. Based on this method, we derive the fixed stepsize Euler scheme for system (2.1), then we give the convergence result of the Euler scheme.

Given a positive integer m, let $h = \frac{\tau}{m}$, and take the grid points for the numerical scheme as follows:

$$\eta_k = \left[\frac{t_k}{h}\right] + 1 - \delta_{\mathbb{Z}}\left(\frac{t_k}{h}\right),$$