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New exact double periodic wave and complex wave solutions for a generalized sinh–Gordon equation

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ABSTRACT

In this paper, dependent and independent variable transformations are introduced to solve a generalized sinh–Gordon equation by using the binary F-expansion method and the knowledge of elliptic equation and Jacobian elliptic functions. Many different new exact solutions such as double periodic wave and complex wave solutions are obtained. Some previous results are extended.

1. Introduction

It is well known that the exact solutions of the sinh–Gordon equations have been extensively studied in the field of theoretical physics (see Refs. [1–7] and references cited therein).

In 2006, Wazwaz [8] studied the following generalized Sinh–Gordon equation:

$$u_{tt} - au_{xx} + b\sinh(nu) = 0, \tag{1}$$

where n is a positive integer and $a,\ b$ are two constants. And he derived families of exact solutions using the reliable tanh method. Tang et al. [9] studied the bifurcation behaviors and exact solutions of the Eq. (1) under three different functions transformations by using the bifurcation theory of dynamical system.

In this paper, we aim to extend the previous works in Refs. [8,9], we shall obtain many new exact solutions of Eq. (1), including double periodic wave and complex wave solutions.

This paper is organized as follows. In Section 2, we introduce the binary F-expansion method briefly. In Section 3, we give many exact solutions of Eq. (1). In Section 4, a short conclusion will be given.

2. The binary F-expansion method

For a given nonlinear partial differential equation

$$\Phi(f(u), u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0, \tag{2}$$

where $f(u)$ is a composite function which is similar to $\sin(nu)$ or $\sinh(nu)$ $(n = 1, 2, \ldots)$ etc. As in Ref. [15], the binary F-expansion method is simply represented as follows:

Step 1: We make a transformation

$$u = \phi\left(\frac{U(\xi)}{V(\eta)}\right), \tag{3}$$

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where $\xi = \lambda_1(x + c_1 t),\ \eta = \lambda_2(x + c_2 t),\ \lambda_1, \lambda_2, c_1, c_2$ are unknown parameters which to be further determined. The transformation $u = \phi\left(\frac{U(\xi)}{V(\eta)}\right)$ was first given by Lamb and used it to solve the sine–Gordon equation [12], $u = \frac{4}{n}\tan^{-1}\left(\frac{U(\xi)}{V(\eta)}\right)$ and $u = \frac{4}{n}\tanh^{-1}\left(\frac{U(\xi)}{V(\eta)}\right)$ are its two special cases. Substituting (3) into (2), yields

$$\Phi(U, U', U'', \ldots, V, V', V'', \ldots) = 0. \tag{4}$$

Step 2: On some constraint conditions, if Eq. (4) can be differentiated as follows

$$U'^2 = P_1 + Q_1 U^2 + R_1 U^4, \tag{5}$$

$$V'^2 = P_2 + Q_2 V^2 + R_2 V^4, \tag{6}$$

where $P_1, Q_1, R_1, P_2, Q_2, R_2$ are some parameters, then, with the aid of Table 1 (see Ref. [14]), we can get the solutions $U(\xi), V(\eta)$ of Eqs. (5) and (6).

Step 3: Substituting the $U(\xi), V(\eta)$ into (3), many exact solutions of Eq. (2) can be obtained.

3. Exact solutions of Eq. (1)

First, let us recall some properties of Jacobian elliptic functions. We know that there exist twelve kinds of Jacobian elliptic functions [10,11]

$$sn(\tau,m), cn(\tau,m), dn(\tau,m), sc(\tau,m), sd(\tau,m), cd(\tau,m),$$
$$ns(\tau,m), nc(\tau,m), nd(\tau,m), cs(\tau,m), ds(\tau,m), dc(\tau,m),$$

where m $(0 < m < 1)$ is a modulus of Jacobian elliptic functions.

When $m \to 1$, the Jacobian functions degenerate to the hyperbolic functions, that is

$$sn(\tau,m) \to \tanh(\tau), cn(\tau,m) \to sech(\tau), dn(\tau,m) \to sech(\tau), sc(\tau,m) \to \sinh(\tau),$$
$$sd(\tau,m) \to \sinh(\tau), cd(\tau,m) \to 1, ns(\tau,m) \to \coth(\tau), nc(\tau,m) \to \cosh(\tau),$$
$$nd(\tau,m) \to \cosh(\tau), cs(\tau,m) \to csch(\tau), ds(\tau,m) \to csch(\tau), dc(\tau,m) \to 1.$$

When $m \to 0$, the Jacobian functions degenerate to the trigonometric functions, i.e.

$$sn(\tau,m) \to \sin(\tau), cn(\tau,m) \to \cos(\tau), dn(\tau,m) \to 1, sc(\tau,m) \to \tan(\tau),$$
$$sd(\tau,m) \to \sin(\tau), cd(\tau,m) \to \cos(\tau), ns(\tau,m) \to \csc(\tau), nc(\tau,m) \to \sec(\tau),$$
$$nd(\tau,m) \to 1, cs(\tau,m) \to \cot(\tau), ds(\tau,m) \to \csc(\tau), dc(\tau,m) \to \sec(\tau).$$

Next, we study Eq. (1). Considering the following transformation:

$$\xi = \lambda(x + ct), \quad \eta = \lambda\left(x + \frac{a}{c}t\right), \quad a \neq c^2, \tag{7}$$

where $\lambda,\ c$ are two parameters to be determined later, Eq. (1) can be rewritten as

$$\lambda^2 c^2 (c^2 - a) u_{\xi\xi} + \lambda^2 a(a - c^2) u_{\eta\eta} + bc^2 \sinh(nu) = 0. \tag{8}$$

By means of a similar ansatz as given in Refs. [12,13], letting

Table 1
Relations between values of (P, Q, R) and corresponding $F(\tau)$ in ODE $F'^2 = P + QF^2 + RF^4$

P	Q	R	$F(\tau)$
1	$-(1+m^2)$	m^2	$sn(\tau,m),\ cd(\tau,m)$
$1-m^2$	$2m^2-1$	$-m^2$	$cn(\tau,m)$
m^2-1	$2-m^2$	-1	$dn(\tau,m)$
m^2	$-(1+m^2)$	1	$ns(\tau,m),\ dc(\tau,m)$
$-m^2$	$2m^2-1$	$1-m^2$	$nc(\tau,m)$
-1	$2-m^2$	m^2-1	$nd(\tau,m)$
1	$2-m^2$	$1-m^2$	$sc(\tau,m)$
1	$2m^2-1$	$-m^2(1-m^2)$	$sd(\tau,m)$
$1-m^2$	$2-m^2$	1	$cs(\tau,m)$
$-m^2(1-m^2)$	$2m^2-1$	1	$ds(\tau,m)$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$ns(\tau,m) \pm cs(\tau,m)$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$nc(\tau,m) \pm sc(\tau,m)$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$ns(\tau,m) \pm ds(\tau,m)$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$sn(\tau,m) \pm ics(\tau,m),\ i^2 = -1$